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Surfaces immersed in $\mathfrak{su}(N+1)$ Lie algebras obtained from the $\mathbb{C}P^N$ sigma models

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Abstract
We study some geometrical aspects of two-dimensional orientable surfaces arising from the study of $\mathbb{C}P^N$ sigma models. To this aim we employ an identification of $\mathbb{R}^{N(N+2)}$ with the Lie algebra $\mathfrak{su}(N+1)$ by means of which we construct a generalized Weierstrass formula for immersion of such surfaces. The structural elements of the surface like its moving frame, the Gauss–Weingarten and the Gauss–Codazzi–Ricci equations are expressed in terms of the solution of the $\mathbb{C}P^N$ model defining it. Further, the first and second fundamental forms, the Gaussian curvature, the mean curvature vector, the Willmore functional and the topological charge of surfaces are expressed in terms of this solution. We present detailed implementation of these results for surfaces immersed in $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ Lie algebras.

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1. Introduction

In this paper we develop further the study of immersions of two-dimensional surfaces in multidimensional Euclidean spaces by means of $\mathbb{C}P^N$ models, which was carried out in a series of papers [17, 19, 20]. The key point is the formulation of the equations defining the immersion directly in the matrix form (cf equation (4.1)) where the immersion takes values in the Lie algebra $\mathfrak{su}(N+1)$, identified by means of the negative of the Killing form with the Euclidean space $\mathbb{R}^{N(N+2)}$. This allows us to formulate explicitly the structural equations for the immersion (the Gauss–Weingarten and the Gauss–Codazzi–Ricci equations) directly in matrix terms. In particular in proposition 3 we establish an explicit form of the Gauss–Weingarten equations satisfied by the moving frame on a surface corresponding to the $\mathbb{C}P^N$ model. This is done in a fashion independent of any specific parametrization. Then we use this result to establish various geometric characteristics of the studied immersions such as curvatures and curvature vectors. All these quantities are directly derived from the map describing the relevant $\mathbb{C}P^N$ model. For the simplest case $N = 1$ equation (4.1) defining the immersion takes the form

$$\mathrm{d}X = \mathrm{i}(\mathrm{d}X_1\sigma_2 + \mathrm{d}X_2\sigma_1 + \mathrm{d}X_3\sigma_3), \tag{1.1}$$

where $\sigma_1, \sigma_2, \sigma_3$ are the usual Pauli matrices, and the differentials of coordinate functions of the immersion $\mathrm{d}X_1, \mathrm{d}X_2, \mathrm{d}X_3$ are given in terms of the affine coordinate W of the $\mathbb{C}P^1$ model by equation (5.3). As follows from equation (5.1) describing the $\mathbb{C}P^1$ model, a particular class of solutions of this model is given by an arbitrary holomorphic function W—in this case the immersion is minimal (i.e., it represents a surface with zero mean curvature) and W expresses the Gauss map of the surface by means of the stereographic projection. This is directly related to the classical Weierstrass–Enneper formulae for an immersion of a minimal surface in $\mathbb{R}^3$. In fact, almost one and half century ago Weierstrass and Enneper showed [11, 45] that every minimal surface in $\mathbb{R}^3$ can be represented locally in terms of two holomorphic functions ψ and ϕ defined on a domain $\mathbb{D} \in \mathbb{C}$ by the following expressions:

$$X(\xi, \bar{\xi}) = \mathrm{Re}\left(\int_0^{\xi} (\psi^2 - \phi^2)\,\mathrm{d}\xi', \mathrm{i}\int_0^{\xi} (\psi^2 + \phi^2)\,\mathrm{d}\xi', -2\int_0^{\xi} \psi\phi\,\mathrm{d}\xi'\right). \tag{1.2}$$

This implies that the complex tangent vector of the immersion is given by

$$\partial X_1 = \psi^2 - \phi^2, \qquad \partial X_2 = \mathrm{i}(\psi^2 + \phi^2), \qquad \partial X_3 = -2\psi\phi, \tag{1.3}$$

where ∂ denotes the (complex) derivative with respect to ξ. Moreover, the metric of the minimal surface is conformal and is expressed in terms of local parameters ξ and $\bar{\xi}$ by the formula

$$\mathrm{d}s^2 = 2(|\psi|^2 + |\phi|^2)^2\,\mathrm{d}\xi\,\mathrm{d}\bar{\xi}. \tag{1.4}$$

This implies, in particular, that the coordinate lines $\mathrm{Re}\,\xi = \text{const}$ and $\mathrm{Im}\,\xi = \text{const}$ describe geodesics on this surface.

The fundamental ideas of Enneper and Weierstrass have since been intensively developed with the purpose of extending this construction to obtain more general types of immersions of surfaces. An interested reader may find a coverage of various stages of the development of the theory in the treatise by Eisenhardt [10] or a survey by Osserman [35] and a contemporary point of view e.g. in recent books by Helein [23, 24], Kenmotsu [28] or Bobenko and Eitner [2], as well as in a number of other places, e.g. [1, 13, 14, 27, 37–39]. An interesting link between the theory of surfaces and infinite-dimensional integrable systems was pursued in a series of papers by Konopelchenko and his collaborators, [4, 30–32], who introduced into considerations a nonlinear Dirac-type system of equations for two complex-valued functions ψ_1 and ψ_2 defined on a domain $\mathbb{D}$ in the complex plane $\mathbb{C}$

$$\partial\psi_1 = (|\psi_1|^2 + |\psi_2|^2)\psi_2, \qquad \bar{\partial}\psi_2 = -(|\psi_1|^2 + |\psi_2|^2)\psi_1. \tag{1.5}$$

In particular, in a paper with Taimanov [32] they showed that for any solutions ψ_1, ψ_2 of the system (1.5) the following integrals over the bilinear combinations of $\psi_i, i = 1, 2$

$$X_1 = \int_\gamma \left(\psi_1^2 - \psi_2^2\right)\mathrm{d}\xi' + \left(\bar{\psi}_1^2 - \bar{\psi}_2{}^2\right)\mathrm{d}\bar{\xi}',$$
$$X_2 = \int_\gamma \left(\psi_1^2 + \psi_2^2\right)\mathrm{d}\xi' - \left(\bar{\psi}_1^2 + \bar{\psi}_2{}^2\right)\mathrm{d}\bar{\xi}', \tag{1.6}$$

$$X_3 = -\int_\gamma \psi_1\psi_2\,\mathrm{d}\xi + \bar{\psi}_1\bar{\psi}_2\,\mathrm{d}\bar{\xi}',$$

may be viewed as giving a parametrization of a surface with a constant mean curvature (CMC), immersed in $\mathbb{R}^3$ by means of the radius vector $X = (X_1, X_2, X_3)$. To see how to reduce more general cases down to this case, cf e.g. [31, 32] and for an intrinsic approach to Dirac operator in the theory of surfaces see [15]. In accordance with [4] we will refer to equations (1.5) and (1.6) as the generalized Weierstrass formulae.

It was shown in our previous work [19] that the generalized Weierstrass formulae for two-dimensional surfaces with non-vanishing mean curvature in multidimensional spaces are equivalent to $\mathbb{C}P^N$ sigma models. This determination has opened a new way for constructing and studying two-dimensional surfaces. The further advantage of use of the $\mathbb{C}P^N$ models in this context lies in the fact that they allow us to replace the methods based on Dirac-type equations by the formalism connected with completely integrable systems, for example Lax pairs, Hamiltonian structures, or systems defining infinite number of conserved quantities. An original procedure for constructing the general classical solutions admitting finite action of the Euclidean two-dimensional $\mathbb{C}P^N$ model was devised by Din and Zakrzewski [8] and followed by a construction by Eells and Wood in [9]. These solutions are obtained by repeated applications of a certain transformation to the basic solution expressed in terms of holomorphic functions. As a result, one gets three classes of solutions: holomorphic, anti-holomorphic and the 'mixed' ones. In this paper we show that to each of these solutions we can associate a surface in $\mathfrak{su}(N+1) \simeq \mathbb{R}^{N(N+2)}$. In the holomorphic (or antiholomorphic) case we are able to integrate completely the equations of the immersion. It turns out that in the $\mathbb{C}P^1$ case the surface is a part of an Euclidean sphere, cf section 5. However for arbitrary $N > 1$ other situations are possible. In example 1 in section 7 we present a one-parameter family of surfaces for which the curvature is not constant but for some specific values of this parameter it reduces to a constant.

The second and third examples in section 7 are concerned with mixed solutions of the $\mathbb{C}P^2$ model. In one case we obtain a surface in $\mathbb{R}^8$, which happens to be immersible in $\mathbb{R}^3$, but the immersion does not come from a $\mathbb{C}P^1$ model, since the curvature is not constant. The other mixed solution leads to a generic surface in $\mathbb{R}^8$ with nonconstant curvatures. All these results raise interesting questions which require further investigations concerning general properties of immersions given by the $\mathbb{C}P^N$ models—either holomorphic or mixed.

Finally, let us note that the outlined approach to the study of surfaces in $\mathbb{R}^m$ lends itself to numerous potential applications. It is useful for description of monodromy of solutions of higher order Painlevé equations and their connection with theory of surfaces. It can also lead to the development of numerical computing tools in the study of surfaces through the techniques of completely integrable systems.

Surfaces immersed in Lie groups, Lie algebras and homogeneous spaces appear in many areas of physics, chemistry and biology [5–7, 33, 34, 36, 40, 42, 43]. The algebraic approach to structural equations of these surfaces has often proved to be very difficult from the computational point of view. A natural geometric approach to derivation and classification of such equations which we propose here seems therefore to be of importance for applications in physics and other sciences.

This paper is organized as follows. In section 2 we introduce basic material on $\mathbb{C}P^N$ models—in presenting it we focus on the use of a compatibility condition, rather than on the usual approach via the Euler–Lagrange equations. The next section is devoted to the presentation of required notions and facts on the structure of complex projective spaces. Here we prove in detail a certain decomposition of the group $\mathbf{SU}(N+1)$, which was previously noted in the paper [41] of Rowe *et al*. In section 3 we show how to use the equations of the $\mathbb{C}P^N$ model to construct an immersion of a surface in the Lie algebra $\mathfrak{su}(N+1)$. The obtained formula extends the classical Weierstrass formula for the conformal immersion of a two-dimensional surface into the three-dimensional Euclidean space. We derive the equations of the moving frame for the above immersion in $\mathfrak{su}(N+1)$ and derive some geometrical characteristics of these surfaces. This analysis is developed further in sections 5 and 6, where, using the conformality of the surfaces obtained from the $\mathbb{C}P^N$ models for $N = 1, 2$, we establish an explicit formula for the moving frame in terms of the data of the model. Let us note that the case $N = 2$ produces surfaces immersed in $\mathbb{R}^8 \simeq \mathfrak{su}(3)$. In section 7 we illustrate our theoretical considerations with some examples based on explicit solutions of the $\mathbb{C}P^2$ sigma model. The last section contains remarks and suggestions regarding possible further developments.

2. Preliminaries on $\mathbb{C}P^N$ models

From a large supply of geometrical models of immersions [21, 22] we concentrate in this paper on a particular class of models, the so-called $\mathbb{C}P^N$ sigma models, see e.g. [48]. The $\mathbb{C}P^N$ sigma model can be defined in terms of functions

$$\mathbb{C} \supset \Omega \ni \xi = \xi^1 + \mathrm{i}\xi^2 \mapsto z = (z_0, z_1, \ldots, z_N) \in \mathbb{C}^{N+1} \tag{2.1}$$

defined on an simply connected domain (i.e., an open connected subset) Ω of the complex plane and satisfying the constraint $z^\dagger \cdot z = 1$. Here and below we employ the standard notation where points of the complex coordinate space $\mathbb{C}^{N+1}$ are denoted by $z = (z_0, z_1, \ldots, z_N)$ and $e_0, e_1, \ldots, e_N$ stand for the standard unit vectors in $\mathbb{C}^{N+1}$ with coordinates $e_{jk} = \delta_{jk}$, for $j, k = 0, \ldots, N$. The standard Hermitian inner product in $\mathbb{C}^{N+1}$, respectively the norm, are denoted by

$$z^\dagger \cdot w = \langle z, w\rangle = \sum_{j=0}^{N} \bar{z}_j w_j, \qquad \text{respectively} \quad |z| = (z^\dagger \cdot z)^{1/2}, \tag{2.2}$$

and the unit sphere in $\mathbb{C}^{N+1}$ corresponding to this norm is

$$S^{2N+1} = \{z \in \mathbb{C}^{N+1} \mid z^\dagger \cdot z = |z|^2 = 1\}.$$

We shall use $\partial_\mu = \partial/\partial\xi^\mu$, $\mu = 1, 2$ to denote ordinary derivatives and D_μ for the covariant derivatives defined according to the formula

$$D_\mu z = \partial_\mu z - z(z^\dagger \cdot \partial_\mu z). \tag{2.3}$$

With this notation the Lagrangian density for such a model is given by (cf e.g. [48])

$$\mathcal{L} = \tfrac{1}{4}(D_\mu z)^\dagger \cdot (D_\mu z), \tag{2.4}$$

and the solutions of the $\mathbb{C}P^N$ model are stationary points of the action functional

$$S = \int_\Omega \mathcal{L}\, \mathrm{d}\xi\, \mathrm{d}\bar{\xi} = \frac{1}{4}\int_\Omega (D_\mu z)^\dagger \cdot (D_\mu z)\, \mathrm{d}\xi\, \mathrm{d}\bar{\xi}. \tag{2.5}$$

The physically relevant case concerns fields which can be extended to the whole Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$, but the case of an arbitrary Ω is also of some interest, especially for questions of (local) differential geometry.

Next we note that since $\mathcal{L}$ is not changed by the transformations $z \mapsto \mathrm{e}^{\mathrm{i}\varphi} z$ with $\varphi \in \mathbb{R}$, it is actually defined by the map $[z] : \Omega \to \mathbb{C}P^N$, where $[z] = \{\mathrm{e}^{\mathrm{i}\varphi} z \mid \varphi \in \mathbb{R}\}$ is the element of the projective space $\mathbb{C}P^N$ corresponding to $z \in S^{2N+1}$. We find it often more convenient to use this latter point of view and describe the model in terms of 'unnormalized' fields $\xi \mapsto f = (f_0, f_1, \ldots, f_N) \in \mathbb{C}^{N+1} \setminus \{0\}$ related to the 'z's' above by

$$z = \frac{f}{|f|}, \qquad \text{where} \quad |f| = (f^\dagger \cdot f)^{1/2}. \tag{2.6}$$

We shall refer to the 'z's' above as inhomogeneous and to the 'f's' as homogeneous coordinates of the model.

Now, using the customary notation for holomorphic and antiholomorphic derivatives

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial \xi^1} - \mathrm{i}\frac{\partial}{\partial \xi^2}\right), \qquad \bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial \xi^1} + \mathrm{i}\frac{\partial}{\partial \xi^2}\right) \tag{2.7}$$

and introducing the orthogonal projector on the orthogonal complement to the complex line in $\mathbb{C}^{N+1}$ determined by f given by

$$P = \mathbf{1}_{N+1} - \frac{1}{f^\dagger \cdot f} f \otimes f^\dagger \tag{2.8}$$

we may express the action functional (2.5) in terms of f's by

$$S = \frac{1}{4}\int_\Omega \frac{1}{f^\dagger \cdot f}(\partial f^\dagger P \bar{\partial} f + \bar{\partial} f^\dagger P \partial f)\,\mathrm{d}\xi\,\mathrm{d}\bar{\xi}. \tag{2.9}$$

Since P is an orthogonal projector, it satisfies

$$P^2 = P, \qquad P^\dagger = P. \tag{2.10}$$

The map $[z]$ is determined by a solution of the Euler–Lagrange equations which is associated with the action (2.9). In terms of homogeneous coordinates f's the equations take the form

$$P\left[\partial\bar{\partial} f - \frac{1}{f^\dagger \cdot f}((f^\dagger \cdot \bar{\partial} f)\partial f + (f^\dagger \cdot \partial f)\bar{\partial} f)\right] = 0. \tag{2.11}$$

Using the projector P we can rewrite (2.11) as

$$[\partial\bar{\partial} P, P] = 0, \tag{2.12}$$

or equivalently as the conservation law

$$\partial[\bar{\partial} P, P] + \bar{\partial}[\partial P, P] = 0. \tag{2.13}$$

Further, introducing the $(N+1) \times (N+1)$ matrix $\mathbb{K}$

$$\mathbb{K} = [\bar{\partial} P, P] = \frac{\bar{\partial} f \otimes f^\dagger - f \otimes \bar{\partial} f^\dagger}{f^\dagger \cdot f} + \frac{f \otimes f^\dagger}{(f^\dagger \cdot f)^2}[(\bar{\partial} f^\dagger \cdot f) - (f^\dagger \cdot \bar{\partial} f)] \tag{2.14}$$

and noting that its Hermitian conjugate is

$$\mathbb{K}^\dagger = -[\partial P, P] = \frac{-\partial f \otimes f^\dagger + f \otimes \partial f^\dagger}{f^\dagger \cdot f} + \frac{f \otimes f^\dagger}{(f^\dagger \cdot f)^2}[(\partial f^\dagger \cdot f) - (f^\dagger \cdot \partial f)], \tag{2.15}$$

we can reformulate (2.13) succinctly as

$$\partial\mathbb{K} - \bar{\partial}\mathbb{K}^\dagger = 0. \tag{2.16}$$

It follows from the above (2.16) that $\partial\mathbb{K}$ is an Hermitian matrix, i.e. $\partial\mathbb{K} \in \mathrm{i}\,\mathfrak{su}(N+1)$.

One can check by a straightforward computation that the complex-valued functions

$$J = \frac{1}{f^\dagger \cdot f}\partial f^\dagger P \partial f, \qquad \bar{J} = \frac{1}{f^\dagger \cdot f}\bar{\partial} f^\dagger P \bar{\partial} f, \tag{2.17}$$

satisfy

$$\bar{\partial} J = 0, \qquad \partial \bar{J} = 0 \tag{2.18}$$

for any solution f of the Euler–Lagrange equations (2.11). Note that J and $\bar{J}$ are invariant under global $\mathbf{U}(N+1)$ transformation, i.e. $f \to \psi f$, $\psi \in \mathbf{U}(N+1)$.

3. Some decompositions of SU($N+1$) and related parametrizations of $\mathbb{C}P^N$

In this section we collect several facts concerning realization of projective space $\mathbb{C}P^N$ as a homogeneous space of the special unitary group $\mathbf{SU}(N+1)$ and discuss related decompositions of the group and its Lie algebra $\mathfrak{su}(N+1)$. The standard reference, where all the details missing here can be found, is the book of Helgason [25].

As is well known, the space $\mathbb{C}P^N$ consists of complex lines through the origin 0 in $\mathbb{C}^{N+1}$ (the one-dimensional complex subspaces of $\mathbb{C}^{N+1}$). We denote by π the map which associates to any nonzero vector $Z = (z_0, \dots, z_N) \in \mathbb{C}^{N+1}$ the line passing through the origin and Z, so that

$$\pi(Z) = \{\lambda Z \mid \lambda \in \mathbb{C}\} = [z_0, z_1, \dots, z_N]. \tag{3.1}$$

The numbers $z_0, z_1, \dots, z_N$, determined up to a nonzero complex number, are called the homogeneous coordinates of the line $\pi(Z)$. The restriction of π to the unit sphere $S^{2N+1} = \{Z \in \mathbb{C}^{N+1} \mid Z^\dagger Z = 1\}$ remains surjective—the resulting map $\pi_H : S^{2N+1} \to \mathbb{C}P^N$ is known as the Hopf fibration. Observe that if the line l passes through the point $Z_0 \in S^{2N+1}$, then the fibre over l, $\pi_H^{-1}(l) = \{Z \in S^{2N+1} | Z = \mathrm{e}^{\mathrm{i}\varphi} Z_0, \varphi \in \mathbb{R}\}$, is just the great circle in S^{2N+1} passing through Z_0.

For any given $j = 0, 1, \dots, N$ one introduces the so-called affine or inhomogeneous coordinates defined in the complement of the set $H_j = \{\pi(Z) \mid Z \in \mathbb{C}_*^{N+1}, z_j = 0\} \subset \mathbb{C}P^N$ by the prescription

$$[z_0, z_1, \dots, z_N] \mapsto \left(\frac{z_0}{z_j}, \dots, \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, \dots, \frac{z_N}{z_j}\right)$$

which sets up a natural isomorphism of $\mathbb{C}^N$ with $\mathbb{C}P^N \setminus H_j$. In the particular case of the affine coordinates defined in the set $U_0 = \mathbb{C}P^N \setminus H_0$ we shall write $W_i = z_i/z_0$.

By transitivity of the action of $\mathbf{SU}(N+1)$ on the set of lines in $\mathbb{C}^{N+1}$ one has a natural identification $G/K_0 \simeq \mathbb{C}P^N$, with K_0 denoting the isotropy group of the standard reference point $l_0 = \pi_H(e_0) = \mathbb{C}e_0$. Now

$$K_0 = \mathbf{S}(\mathbf{U}(1) \times \mathbf{U}(N)) = \left\{ \begin{pmatrix} (\det \mathbf{A})^{-1} & 0 \\ 0 & \mathbf{A} \end{pmatrix} \middle| \mathbf{A} \in \mathbf{U}(N) \right\} \tag{3.2}$$

and the identification above is written as $\mathbb{C}P^N \simeq \mathbf{SU}(N+1)/\mathbf{S}(\mathbf{U}(1) \times \mathbf{U}(N))$. Passing to the Lie algebra level and denoting the respective Lie algebras by $\mathfrak{g} = \mathfrak{su}(N+1)$ and

$$\mathfrak{k}_0 = \left\{ \begin{pmatrix} -\mathrm{tr}\,\mathbf{A} & 0 \\ 0 & \mathbf{A} \end{pmatrix} \middle| \mathbf{A} \in \mathfrak{u}(N) \right\}$$

one has the direct sum decomposition of the isotropy Lie algebra

$$\mathfrak{k}_0 = \mathfrak{c} \oplus \mathfrak{su}(N), \qquad \text{where} \quad \mathfrak{c} = \left\{ \begin{pmatrix} -\mu & 0 \\ 0 & \frac{\mu}{N}\mathbf{1}_N \end{pmatrix} \middle| \mu \in \mathrm{i}\mathbb{R} \right\} \simeq \mathrm{i}\mathbb{R}$$

with $\mathfrak{c}$ being its centre. To study other decompositions we first recall that the Killing form of $\mathfrak{g} = \mathfrak{su}(N+1)$ is given by the formula

$$B(X, Y) = 2(N+1)\,\mathrm{tr}(XY) \tag{3.3}$$

and is negative definite. The space $\mathfrak{su}(N+1)$ of skew-Hermitian matrices can thus be given the structure of a real Euclidean space of dimension $N(N+2)$ by taking the negative of the

Killing form as the inner product. The orthogonal complement to $\mathfrak{k}_0$ with respect to this inner product consists of matrices of the form

$$Z(x) = \begin{pmatrix} 0 & -x^\dagger \\ x & \mathbf{0}_N \end{pmatrix} = x \otimes e_0^\dagger - e_0 \otimes x^\dagger, \tag{3.4}$$

where $x = (x_1, \ldots, x_N) \in \mathbb{C}^N$ and $\mathbf{0}_N$ is the $N \times N$ zero matrix, and this yields the orthogonal decomposition $\mathfrak{g} = \mathfrak{k}_0 \oplus \mathfrak{p}$.

This later fact is a starting point for introducing a useful parametrization of the projective space, analogous to the spherical coordinates on the Euclidean sphere. Observe that the adjoint action of the isotropy group K_0 on $\mathfrak{p}$ reduces to the action of $\mathbf{U}(N)$ on $\mathbb{C}^N$ given by the following formula:

$$\begin{pmatrix} (\det A)^{-1} & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} 0 & -x^\dagger \\ x & \mathbf{0}_N \end{pmatrix} \begin{pmatrix} \det A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} 0 & -(\det A)^{-1}(Ax)^\dagger \\ \det(A)Ax & \mathbf{0}_N \end{pmatrix}.$$

The action is clearly transitive on the unit sphere in $\mathfrak{p}$ and essentially this fact implies validity of the next result (for a general form of such decompositions cf [25, p 402]). To formulate it we first introduce more notations. Set $H = Z(e_1)$ and let $\mathfrak{a} = \mathbb{R}H \subset \mathfrak{p}$. The Lie subgroup $\exp \mathfrak{a} = \{\exp \alpha H \mid \alpha \in \mathbb{R}\}$ of $\mathbf{SU}(N+1)$ consists of matrices

$$\exp \alpha H = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & \mathbf{1}_{N-1} \end{pmatrix} = \begin{pmatrix} \mathbf{R}(\alpha) & 0 \\ 0 & \mathbf{1}_{N-1} \end{pmatrix}, \tag{3.5}$$

where we have set $\mathbf{R}(\alpha) = \left(\begin{smallmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{smallmatrix}\right)$ and which is isomorphic to $\mathbf{SO}(2)$. The corresponding maximal torus in $\mathbb{C}P^N$ is

$$A = (\exp \mathfrak{a}) \cdot l_0 = \{[\cos\alpha, \sin\alpha, 0, \ldots, 0] \mid \alpha \in \mathbb{R}\}. \tag{3.6}$$

Denoting further by $M \subset K_0$ the centralizer and by $M' \subset K_0$ the normalizer of $\mathfrak{a}$ in K_0, i.e. $M = \{k \in K_0 \mid kHk^{-1} = H\}$ and $M' = \{k \in K_0 \mid kHk^{-1} \subset \mathbb{R}H\}$, we see that

$$M = \left\{ \begin{pmatrix} u & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & U \end{pmatrix} \;\middle|\; U \in \mathbf{U}(N-1), u \in \mathbf{U}(1), u^2 \det U = 1 \right\} \tag{3.7}$$

$$M' = \left\{ \begin{pmatrix} u & 0 & 0 \\ 0 & \varepsilon u & 0 \\ 0 & 0 & \varepsilon U \end{pmatrix} \;\middle|\; U \in \mathbf{U}(N-1), u \in \mathbf{U}(1), \varepsilon = \pm 1, u^2 \det U = 1 \right\}. \tag{3.8}$$

The factor group $M'/M = \mathbb{Z}_2$ is the Weyl group associated with $\mathbb{C}P^N$.

We need one more notation. Given $k \leqslant n$ complex numbers of modulus 1, $\lambda_1, \ldots, \lambda_k$, we denote by $\delta(\lambda_1, \ldots, \lambda_k)$ the $k \times k$ diagonal matrix with $\lambda_1, \ldots, \lambda_k$ along the diagonal,

$$\delta(\lambda_1, \ldots, \lambda_k) = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \ldots & \lambda_k \end{pmatrix}. \tag{3.9}$$

We can now formulate the main result of this section, which will be used extensively later on. It describes a certain decomposition of the group $\mathbf{SU}(N+1)$ related to the spherical parametrization of the projective space which is stated as the point (c) of the proposition below. The points (a) and (b) are included for readers' convenience and comprise the classical decompositions which can be found in general form e.g. in [25, p 402]). It should be pointed out that (c) is an elaboration of a result stated in [41], but proved there only for $\mathbf{SU}(3)$.

Proposition 1 (polar decompositions). *Let* $G = \mathbf{SU}(N+1)$, $K_0 = \mathbf{S}(\mathbf{U}(1) \times \mathbf{U}(N))$, *and let the remaining notations be as explained above.*

(a) Every element of G *can be written as a product* $g = k(\exp\theta H)k'$ *with* $k, k' \in K_0$. *More precisely, the map*

$$K_0 \times \exp\mathfrak{a} \times K_0 \ni (k, \exp\theta H, k') \mapsto k(\exp\theta H)k' \in G \tag{3.10}$$

arising from the group multiplication is a smooth surjection.

(b) The map

$$K_0/M \times A \ni (kM, \exp\theta H \cdot l_0) \mapsto k \cdot \exp\theta H \cdot l_0 \in \mathbb{C}P^N \tag{3.11}$$

is a smooth surjection and is a double covering on the complement of the point $l_0 \in \mathbb{C}P^N$.

(c) Consider $\mathbf{SU}(N)$ *as a subgroup of* K_0 *by means of the injection* $\mathbf{A} \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & \mathbf{A} \end{smallmatrix}\right)$ *and let* $\mathbf{U}(1)$ *be diagonally embedded into* $\mathbf{SU}(2)$ *by means of*

$$\delta : \mathbf{U}(1) \ni \mu \mapsto \delta(\mu, \overline{\mu}) = \begin{pmatrix} \mu & 0 \\ 0 & \overline{\mu} \end{pmatrix}. \tag{3.12}$$

Then the map

$$\begin{aligned} &\mathbf{SU}(N) \times \mathbf{U}(1) \times \mathbf{SO}(2) \times \mathbf{SU}(N) \longrightarrow \mathbf{SU}(N+1) \\ &\qquad (\mathbf{A}_1, \mu, \mathbf{R}(\theta), \mathbf{A}_2) \mapsto \\ &\begin{pmatrix} 1 & 0 \\ 0 & \mathbf{A}_1 \end{pmatrix} \cdot \begin{pmatrix} \delta(\mu,\overline{\mu}) & 0 \\ 0 & \mathbf{1}_{N-1} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{R}(\theta) & 0 \\ 0 & \mathbf{1}_{N-1} \end{pmatrix} \cdot \begin{pmatrix} \delta(\mu,\overline{\mu}) & 0 \\ 0 & \mathbf{1}_{N-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{A}_2 \end{pmatrix} \end{aligned} \tag{3.13}$$

is a smooth surjection.

Proof. We are going to prove only part (c) of the statement and this follows by simple matrix calculations from the polar decomposition given in equation (3.10). Assume that we have a product of the form

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \mathbf{A}_1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{R}(\theta) & 0 \\ 0 & \mathbf{1}_{N-1} \end{pmatrix} \cdot \begin{pmatrix} \lambda_2 & 0 \\ 0 & \mathbf{A}_2 \end{pmatrix}, \tag{3.14}$$

where $\mathbf{A}_i \in \mathbf{U}(N)$ and $\lambda_i \det \mathbf{A}_i = 1$ for $i = 1, 2$. By splitting a factor of the form

$$\begin{pmatrix} \delta(\lambda, \lambda, \lambda^{-2}) & 0 \\ 0 & \mathbf{1}_{N-2} \end{pmatrix}, \qquad \lambda = \lambda_1^{1/2}\lambda_2^{-1/2}$$

from the matrix on the left-hand side in this product and commuting it with the middle term, we can bring the entries in the top left corners of the matrices on both sides of equation (3.14) to coincide with each other, thus obtaining the product

$$\begin{pmatrix} \mu & 0 \\ 0 & \mathbf{A}_1' \end{pmatrix} \cdot \begin{pmatrix} \mathbf{R}(\theta) & 0 \\ 0 & \mathbf{1}_{N-1} \end{pmatrix} \cdot \begin{pmatrix} \mu & 0 \\ 0 & \mathbf{A}_2' \end{pmatrix}, \qquad \mu = (\lambda_1\lambda_2)^{1/2}$$

with the relations $\mu \det \mathbf{A}'_i = 1$ for $i = 1, 2$ still satisfied. Now it remains only to split off the factors of the form $\left(\begin{smallmatrix} \delta(\mu,\overline{\mu}) & 0 \\ 0 & \mathbf{1}_{N-1} \end{smallmatrix}\right)$ from the both extreme terms to get the sought for form (3.13) of the product. $\square$

Remark 1. The statement in (c) above is simply that every element of $\mathbf{SU}(N+1)$ can be written as a product of four matrices belonging to the above given subgroups. Writing down the product of the middle terms in equation (3.13) explicitly we obtain

$$\begin{pmatrix}\delta(\mu,\overline{\mu}) & 0\\ 0 & \mathbf{1}_{N-1}\end{pmatrix}\cdot\begin{pmatrix}\mathbf{R}(\theta) & 0\\ 0 & \mathbf{1}_{N-1}\end{pmatrix}\cdot\begin{pmatrix}\delta(\mu,\overline{\mu}) & 0\\ 0 & \mathbf{1}_{N-1}\end{pmatrix}$$
$$= \begin{pmatrix}\mu^2\cos\theta & -\sin\theta & 0 & \dots & 0\\ \sin\theta & \mu^{-2}\cos\theta & 0 & \dots & 0\\ 0 & 0 & & & \\ \vdots & \vdots & & \mathbf{1}_{N-1} & \\ 0 & 0 & & & \end{pmatrix} \tag{3.15}$$

so that the right-hand side of the decomposition (3.13) reduces to

$$\begin{pmatrix}1 & 0 & \dots & 0\\ 0 & & & \\ \vdots & & \mathbf{A}_1 & \\ 0 & & & \end{pmatrix}\begin{pmatrix}\mu^2\cos\theta & -\sin\theta & 0 & \dots & 0\\ \sin\theta & \mu^{-2}\cos\theta & 0 & \dots & 0\\ 0 & 0 & & & \\ \vdots & \vdots & & \mathbf{1}_{N-1} & \\ 0 & 0 & & & \end{pmatrix}\begin{pmatrix}1 & 0 & \dots & 0\\ 0 & & & \\ \vdots & & \mathbf{A}_2 & \\ 0 & & & \end{pmatrix}, \tag{3.16}$$

where $\mathbf{A}_1, \mathbf{A}_2 \in \mathbf{SU}(N)$, $\mu = \mathrm{e}^{\mathrm{i}\alpha} \in \mathbf{U}(1)$ and $\theta, \alpha \in \mathbb{R}$, which up to unimportant changes in parametrization, is the expression given in [41, equation (2) on p 3605].

The polar decomposition (3.10) reduces to the following decomposition of $\mathbf{SU}(3)$ which appears to be known—it can be found e.g. in [26] and [41].

Corollary 1. *Each element of the* $\mathbf{SU}(3)$ *group can be decomposed into the following product:*

$$g = \begin{pmatrix}1 & 0\\ 0 & U_1\end{pmatrix}\begin{pmatrix}\delta(\lambda,\overline{\lambda}) & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}R(\alpha) & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\delta(\lambda,\overline{\lambda}) & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & U_2\end{pmatrix}, \tag{3.17}$$

where $U_i \in \mathbf{SU}(2)$ *for* $i = 1, 2$ *and*

$$R(\alpha) = \begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix} \qquad \text{and} \qquad \delta(\lambda) = \begin{pmatrix}\lambda & 0\\ 0 & \overline{\lambda}\end{pmatrix}$$

with $\lambda \in \mathbb{C}$ *with* $|\lambda| = 1$. *Writing this in a more explicit fashion we have*

$$g = \begin{pmatrix}1 & 0 & 0\\ 0 & a_1 & b_1\\ 0 & -\overline{b_1} & \overline{a_1}\end{pmatrix}\begin{pmatrix}\lambda^2\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \lambda^{-2}\cos\alpha & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & a_2 & b_2\\ 0 & -\overline{b_2} & \overline{a_2}\end{pmatrix}, \tag{3.18}$$

where $a_i, b_i \in \mathbb{C}$ *for* $i = 1, 2$ *satisfy* $|a_i|^2 + |b_i|^2 = 1$ *and* $\lambda \in \mathbb{C}$ *is of modulus 1;* $|\lambda| = 1$.

We finish this section by presenting explicit orthogonal bases for the Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$, which will be used in our future discussion of the $\mathbb{C}P^1$ and $\mathbb{C}P^2$ models. For uniformity we use in this case the inner product

$$(X, Y) = -\tfrac{1}{2}\,\mathrm{tr}(XY), \tag{3.19}$$

rather than the Killing form given by the formula

$$B(X, Y) = \begin{cases}4\,\mathrm{tr}(XY), & N = 1\\ 6\,\mathrm{tr}(XY), & N = 2\end{cases} \qquad X, Y \in \mathfrak{su}(N+1)\,.$$

An orthonormal basis is given for the case $N = 1$ by the matrices $-\mathrm{i}\sigma_j$, for $j = 1, 2, 3$, where σ_j denote the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.20}$$

Now, $\mathfrak{su}(3)$ is eight-dimensional (over $\mathbb{R}$) and consists of matrices of the form

$$\begin{pmatrix} \mathrm{i}a_1 & z_0 & z_1 \\ -\bar{z}_0 & \mathrm{i}a_2 & z_2 \\ -\bar{z}_1 & -\bar{z}_2 & \mathrm{i}a_3 \end{pmatrix}, \qquad \text{with} \quad z_0, z_1, z_2 \in \mathbb{C}, a_1, a_2, a_3 \in \mathbb{R}, a_1 + a_2 + a_3 = 0.$$

For this case we shall choose a basis adapted to the decomposition $\mathfrak{g} = \mathfrak{k}_0 \oplus \mathfrak{p}$, where the isotropy Lie subalgebra $\mathfrak{k}_0$ is given by

$$\mathfrak{k}_0 = \left\{ \begin{pmatrix} \mathrm{i}a_0 & 0 & 0 \\ 0 & \mathrm{i}a_1 & -\bar{z} \\ 0 & z & \mathrm{i}a_2 \end{pmatrix} \middle| \; a_0 + a_1 + a_2 = 0, a_j \in \mathbb{R}, z \in \mathbb{C} \right\} \tag{3.21}$$

and may be further decomposed as

$$\mathfrak{k}_0 = \mathfrak{u}(1) \oplus \mathfrak{u}(2)\,. \tag{3.22}$$

Accordingly, as its orthogonal basis we take $\{S_j \mid j = 1, \ldots, 4\}$, where

$$S_j = \begin{pmatrix} 0 & 0 \\ 0 & -\mathrm{i}\sigma_j \end{pmatrix}, \qquad j = 1, 2, 3, \qquad S_4 = \begin{pmatrix} -2\mathrm{i} & 0 & 0 \\ 0 & \mathrm{i} & 0 \\ 0 & 0 & \mathrm{i} \end{pmatrix}. \tag{3.23}$$

$\mathfrak{p}$, which is the orthogonal complement of $\mathfrak{k}_0$, consists of matrices defined in (3.4) which have the form

$$Z(x) = \begin{pmatrix} 0 & -\bar{x}_1 & -\bar{x}_2 \\ x_1 & 0 & 0 \\ x_2 & 0 & 0 \end{pmatrix}, \qquad \text{where} \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{C}^2.$$

We supplement the above defined basis of $\mathfrak{k}_0$ by the following basis for $\mathfrak{p}$

$$S_5 = Z(e_1) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad S_6 = Z(e_2) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{3.24}$$

$$S_7 = Z(\mathrm{i}e_1) = \begin{pmatrix} 0 & \mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad S_8 = Z(\mathrm{i}e_2) = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}. \tag{3.25}$$

and observe that these bases are orthogonal with respect to the Killing form. Thus the matrices $\{S_1, \ldots, S_8\}$ form an orthogonal basis for $\mathfrak{su}(3)$.

4. The structural equations of surfaces immersed in $\mathfrak{su}(N+1)$ algebras obtained through $\mathbb{C}P^N$ models

In order to investigate immersions defined by means of solutions of the $\mathbb{C}P^N$ models and, in particular, to envisage the moving frames and the corresponding Gauss–Weingarten and the

Gauss–Codazzi–Ricci equations, we use the Euclidean structure of the $\mathfrak{su}(N+1)$ Lie algebra, which we identify with $\mathbb{R}^{N(N+2)}$ in the way described in the previous section. However, our considerations do not necessitate in using any special privileged system of coordinates.

Let us assume that the matrix $\mathbb{K}$ given by (2.14) is constructed from a solution f of the Euler–Lagrange equation (2.11) defined on an open connected and simply connected set $\Omega \subset \mathbb{C}$. The conservation law (2.16) then holds and implies that the matrix-valued 1-form

$$\mathrm{d}X = \mathrm{i}(\mathbb{K}^\dagger\,\mathrm{d}\xi + \mathbb{K}\,\mathrm{d}\bar{\xi}) = \mathrm{i}(\mathbb{K}^\dagger + \mathbb{K})\,\mathrm{d}\xi^1 - (\mathbb{K}^\dagger - \mathbb{K})\,\mathrm{d}\xi^2 \tag{4.1}$$

is closed ($\mathrm{d}(\mathrm{d}X) = 0$) and takes values in the Lie algebra $\mathfrak{su}(N+1)$ of anti-Hermitian matrices. By decomposing $\mathrm{d}X$ into the real and imaginary parts we write

$$\mathrm{d}X = \mathrm{d}X^1 + \mathrm{i}\mathrm{d}X^2, \tag{4.2}$$

where the 1-forms $\mathrm{d}X^1$ and $\mathrm{d}X^2$ with values in $\mathfrak{sl}(N+1,\mathbb{R})$ are anti-symmetric and symmetric, respectively, i.e.

$$(\mathrm{d}X^1)^T = -\mathrm{d}X^1, \qquad (\mathrm{d}X^2)^T = \mathrm{d}X^2,$$

with the superscript T denoting the transposition. From the closedness of the 1-form $\mathrm{d}X$ it follows that the integral

$$\mathrm{i}\int_\gamma (\mathbb{K}^\dagger\,\mathrm{d}\xi + \mathbb{K}\,\mathrm{d}\bar{\xi}) = X(\xi,\bar{\xi}) \tag{4.3}$$

locally depends only on the end points of the curve γ (i.e., it is locally independent of the trajectory in the complex plane $\mathbb{C}$). The integral defines a mapping

$$X : \Omega \ni (\xi,\bar{\xi}) \mapsto X(\xi,\bar{\xi}) \in \mathfrak{su}(N+1) \simeq \mathbb{R}^{N(N+2)} \tag{4.4}$$

which will be called the generalized Weierstrass formula for the immersion. The tangent vectors of this immersion, by virtue of (4.1), are

$$\partial_1 X = \mathrm{i}(\mathbb{K}^\dagger + \mathbb{K}), \qquad \partial_2 X = -(\mathbb{K}^\dagger - \mathbb{K}) \tag{4.5}$$

and the complex tangent vectors are

$$\partial X = \mathrm{i}\mathbb{K}^\dagger, \quad \bar{\partial} X = \mathrm{i}\mathbb{K}. \tag{4.6}$$

Hence a surface $\mathbb{F}$ associated with the $\mathbb{C}P^N$ model (2.11) by means of the immersion (4.4) satisfies the following

Proposition 2 (metric form)**.** *Components of the metric form induced on $\mathbb{F}$ by the Euclidean structure in $\mathfrak{su}(N+1)$ defined by the negative of the Killing form (3.3) are given by*

$$\begin{aligned} g_{11} &= (\partial_1 X, \partial_1 X) = 2(N+1)\,\mathrm{tr}(\mathbb{K}^\dagger + \mathbb{K})^2 \\ g_{22} &= (\partial_2 X, \partial_2 X) = -2(N+1)\,\mathrm{tr}(\mathbb{K}^\dagger - \mathbb{K})^2 \\ g_{12} &= (\partial_1 X, \partial_2 X) = 2i(N+1)\,\mathrm{tr}[(\mathbb{K}^\dagger + \mathbb{K})(\mathbb{K}^\dagger - \mathbb{K})]. \end{aligned} \tag{4.7}$$

The components of the metric form with respect to the complex tangent vectors are given by the following expressions:

$$g_{\xi,\xi} = (\partial X, \partial X) = J, \qquad g_{\bar{\xi},\bar{\xi}} = (\bar{\partial} X, \bar{\partial} X) = \bar{J}, \qquad g_{\xi,\bar{\xi}} = g_{\bar{\xi},\xi} = (\partial X, \bar{\partial} X) = q,$$

where J and $\bar{J}$ are functions defined by (2.17) and q is a non-negative (real-valued) function given by

$$q = \frac{1}{f^\dagger f}\bar{\partial} f^\dagger P \partial f \geqslant 0.$$

Consequently the first fundamental form $\mathbf{I}$ *of the surface* $\mathbb{F}$ *is given with respect to the complex coordinates* $\xi, \bar{\xi}$ *by*

$$\mathbf{I} = J\,\mathrm{d}\xi^2 + 2q\,\mathrm{d}\xi\,\mathrm{d}\bar{\xi} + \bar{J}\,\mathrm{d}\bar{\xi}^2. \tag{4.8}$$

In section 5 we compute explicitly coefficients of the metric form in the case of a $\mathbb{C}P^1$ model.

As usual, we denote

$$g = g_{\xi,\xi}g_{\bar{\xi},\bar{\xi}} - g^2_{\xi,\bar{\xi}} = 4(q^2 - |J|^2)$$

the determinant of the metric form. It is known that the Gaussian curvature of the surface $\mathbb{F}$ with respect to the induced metric is given by

$$\mathbf{K} = \frac{1}{2\sqrt{g}}\bar{\partial}\left[\frac{1}{\sqrt{g}}(-2\partial q + q\partial(\ln J))\right]. \tag{4.9}$$

The quantity $J\,\mathrm{d}\xi^2$ defined on $\mathbb{F}$, called Hopf differential, is invariant with respect to conformal changes of coordinates. We use this freedom to simplify the corresponding equations.

Our next task is to determine a moving frame on the surface $\mathbb{F}$ and to write the corresponding Gauss–Weingarten equations expressed in terms of a solution f satisfying the $\mathbb{C}P^N$ sigma model equations (2.11). Using the Gram–Schmidt orthogonalization procedure to construct and write explicitly expressions for the normals η_k to a given surface in $\mathbb{R}^{N(N+2)}$ can, in general, be a complicated task. An alternative way we propose here involves the use of the isomorphism of $\mathbb{R}^{N(N+2)}$ with the Lie algebra $\mathfrak{su}(N+1)$. In this representation, the equations for a moving frame on the surface can be written in terms of $(N+1)\times(N+1)$ skew-Hermitian matrices. To simplify, in the following calculations we suppress the normalization factor $2(N+1)$ in the definition of the inner product in $\mathfrak{su}(N+1)$—cf (3.3). We introduce real normal vectors $\eta_3, \ldots, \eta_{N(N+2)}$ to the surface $\mathbb{F}$ and consider the frame

$$\eta = (\eta_1 = \partial X, \eta_2 = \bar{\partial}X, \eta_3, \ldots, \eta_{N(N+2)})^T$$

with components satisfying the following conditions:

$$\begin{aligned} &(\partial X, \eta_k) = 0, && (\bar{\partial}X, \eta_k) = 0, \\ &(\eta_j, \eta_k) = \delta_{jk}, && j, k = 3, \ldots, N(N+2). \end{aligned} \tag{4.10}$$

Next we define

$$J_k = \mathrm{tr}(\partial^2 X\cdot\eta_k), \qquad H_k = \mathrm{tr}(\partial\bar{\partial}X\cdot\eta_k). \tag{4.11}$$

Now we can formulate the following

Proposition 3 (the structural equations)**.** *For any solution* f *of the* $\mathbb{C}P^N$ *sigma model equations (2.11), such that the determinant of the induced metric* g *is nonzero in some neighbourhood of a regular point* $p = (\xi_0, \bar{\xi}_0)$ *in* $\mathbb{C}$*, there exists in this neighbourhood a moving frame* η *on this surface which satisfies the following Gauss–Weingarten equations:*

$$\partial\eta_i = A_{il}\eta_l, \qquad \bar{\partial}\eta_i = B_{il}\eta_l, \qquad i, l = 1, \ldots, N(N+2), \tag{4.12}$$

where the $N(N+2)$ *by* $N(N+2)$ *matrices* A *and* B *have the form*

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & J_3 & \cdots & J_{N(N+2)} \\ 0 & 0 & H_3 & \cdots & H_{N(N+2)} \\ \alpha_{1,3} & \beta_{1,3} & 0 & \cdots & S_{3,N(N+2)} \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_{1,N(N+2)} & \beta_{1,N(N+2)} & -S_{3,N(N+2)} & \cdots & 0 \end{pmatrix}, \tag{4.13}$$

and

$$B = \begin{pmatrix} 0 & 0 & H_3 & \cdots & H_{N(N+2)} \\ a_{2,1} & a_{2,2} & \bar{J}_3 & \cdots & \bar{J}_{N(N+2)} \\ \alpha_{2,3} & \beta_{2,3} & 0 & & \bar{S}_{3,N(N+2)} \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_{2,N(N+2)} & \beta_{2,N(N+2)} & -\bar{S}_{3,N(N+2)} & \cdots & 0 \end{pmatrix}. \tag{4.14}$$

The elements of A and B take the form

$$\begin{aligned} &S_{jk} + S_{kj} = 0, \qquad \bar{S}_{jk} + \bar{S}_{kj} = 0, \qquad j \neq k \\ &\alpha_{1,j} = \frac{1}{g}(H_j g_{\xi,\bar{\xi}} - J_j g_{\bar{\xi},\bar{\xi}}), \qquad \beta_{1,j} = \frac{1}{g}(J_j g_{\xi,\bar{\xi}} - H_j g_{\xi,\xi}), \\ &\alpha_{2,j} = \frac{1}{g}(\bar{J} g_{\xi,\bar{\xi}} - H_j g_{\bar{\xi},\bar{\xi}}), \qquad \beta_{2,j} = \frac{1}{g}(H_j g_{\xi,\bar{\xi}} - \bar{J}_j g_{\xi,\xi}), \end{aligned} \tag{4.15}$$

where

$$\begin{aligned} a_{1,1} = -a_{2,2} &= \frac{1}{g}\,\mathrm{Re}\left\{\frac{1}{f^\dagger f}(\bar{J}\partial f^\dagger + g_{\xi,\bar{\xi}}\bar{\partial} f^\dagger)P\partial^2 f - \frac{2\partial f^\dagger f}{(f^\dagger f)^2}(\bar{\partial} f^\dagger P \partial f)g_{\xi,\bar{\xi}} - \frac{2f^\dagger \partial f}{f^\dagger f}|J|^2\right\}, \\ a_{1,2} &= \frac{1}{g}\,\mathrm{Re}\left\{\frac{-1}{f^\dagger f}(J\bar{\partial} f^\dagger + g_{\xi,\bar{\xi}}\partial f^\dagger)P\partial^2 f + \frac{2\partial f^\dagger f}{(f^\dagger f)^2}(\bar{\partial} f^\dagger P\partial f)\bar{J} + \frac{2f^\dagger\partial f}{f^\dagger f}J g_{\xi,\bar{\xi}}\right\}, \\ a_{2,1} &= \frac{1}{g}\,\mathrm{Re}\left\{\frac{1}{f^\dagger f}(J\bar{\partial} f^\dagger + g_{\xi,\bar{\xi}}\partial f^\dagger)P\bar{\partial}^2 f - \frac{2\bar{\partial} f^\dagger f}{(f^\dagger f)^2}(\partial f^\dagger P\bar{\partial} f)g_{\xi,\bar{\xi}} - \frac{2f^\dagger\bar{\partial} f}{f^\dagger f}|J|^2\right\}. \end{aligned} \tag{4.16}$$

The Gauss–Codazzi–Ricci equations are given by

$$\bar{\partial} A - \partial B + [A, B] = 0 \tag{4.17}$$

and coincide with the equations of the $\mathbb{C}P^N$ *sigma model (2.11).*

We note that the elements $a_{i,j}$ are the usual Christoffel symbols.

Proof. Note that for any solution of the $\mathbb{C}P^N$ equations (2.11) the matrices ∂X and $\bar{\partial} X$ are defined by (4.6). As can be checked by a straightforward computation using (2.11), the mixed derivatives $\partial\bar{\partial} X$ and $\bar{\partial}\partial X$ coincide and are normal to the surface

$$\begin{aligned} \partial\bar{\partial} X &= [\partial P, \bar{\partial} P] \\ &= \frac{1}{f^\dagger f}(P\partial f \otimes \bar{\partial} f^\dagger P - P\bar{\partial} f \otimes \partial f^\dagger P) + \frac{1}{(f^\dagger f)^2}(\partial f^\dagger P\bar{\partial} f - \bar{\partial} f^\dagger P\partial f) f \otimes f^\dagger \\ &= -[\bar{\partial} P, \partial P] = \bar{\partial}\partial X. \end{aligned} \tag{4.18}$$

Combining this equation with the $\mathbb{C}P^N$ equations (2.11), expressed in terms of the projector P, we obtain

$$\begin{aligned} \mathrm{tr}(\partial\bar{\partial} X \cdot \partial X) = \mathrm{tr}([\partial P, \bar{\partial} P] \cdot [\partial P, P]) = 0, \\ \mathrm{tr}(\partial\bar{\partial} X \cdot \bar{\partial} X) = \mathrm{tr}([\partial P, \bar{\partial} P] \cdot [\bar{\partial} P, P]) = 0. \end{aligned} \tag{4.19}$$

As a direct consequence of differentiation of the normals (4.10) we get

$$\begin{aligned} &(\partial\eta_j, \eta_k) + (\partial\eta_k, \eta_j) = S_{jk} + S_{kj} = 0, \\ &(\bar{\partial}\eta_j, \eta_k) + (\bar{\partial}\eta_k, \eta_j) = \bar{S}_{jk} + \bar{S}_{kj} = 0, \qquad j \neq k \end{aligned} \tag{4.20}$$

and

$$\begin{aligned} &(\bar{\partial}\eta_j, \partial X) + (\eta_j, \partial\bar{\partial} X) = 0, \qquad (\bar{\partial}\eta_j, \bar{\partial} X) + (\eta_j, \bar{\partial}^2 X) = 0, \\ &(\partial\eta_j, \partial X) + (\eta_j, \partial^2 X) = 0, \qquad (\partial\eta_j, \bar{\partial} X) + (\eta_j, \bar{\partial}\partial X) = 0. \end{aligned} \tag{4.21}$$

Using the expressions (4.12)–(4.14) and (2)) we come to the following set of linear equations

$$\begin{aligned}
&g_{\xi,\bar\xi}\alpha_{1,j} + g_{\bar\xi,\bar\xi}\beta_{1,j} + J_j = 0,\\
&g_{\xi,\xi}\alpha_{1,j} + g_{\xi,\bar\xi}\beta_{1,j} + H_j = 0, \qquad j = 3, \ldots, N(N+2)\\
&g_{\xi,\bar\xi}\alpha_{2,j} + g_{\bar\xi,\bar\xi}\beta_{2,j} + \bar J_j = 0,\\
&g_{\xi,\xi}\alpha_{2,j} + g_{\xi,\bar\xi}\beta_{2,j} + H_j = 0,
\end{aligned} \tag{4.22}$$

which allow us to determine elements $\alpha_{i,j}$ and $\beta_{i,j}$ in terms of the coefficients of the metric form, H_j and J_j. As can be easily calculated, they take the form (4.15) asserted in proposition 3. The second derivatives $\partial^2 X$ and $\bar\partial^2 X$ are

$$\begin{aligned}
\partial^2 X &= \frac{1}{f^\dagger f}(P\partial^2 f \otimes f^\dagger - f \otimes \partial^2 f P)\\
&\qquad + \frac{2}{(f^\dagger f)^2}[(\partial f^\dagger \cdot f) f \otimes \partial f^\dagger P - (f^\dagger \cdot \partial f) P \partial f \otimes f^\dagger],\\
\bar\partial^2 X &= \frac{1}{f^\dagger f}(f \otimes \bar\partial^2 f^\dagger P - P\bar\partial^2 f \otimes f^\dagger)\\
&\qquad + \frac{2}{(f^\dagger f)^2}[(f^\dagger \cdot \bar\partial f) P \bar\partial f \otimes f^\dagger - (\bar\partial f^\dagger \cdot f) f \otimes \bar\partial f^\dagger P].
\end{aligned} \tag{4.23}$$

Let us observe that the following traces (and their complex conjugates) vanish:

$$\begin{aligned}
&\operatorname{tr}((\partial^2 X - a_{1,1}\partial X - a_{1,2}\bar\partial X) \cdot \partial X) = 0,\\
&\operatorname{tr}((\partial^2 X - a_{2,1}\partial X - a_{2,2}\bar\partial X) \cdot \bar\partial X) = 0.
\end{aligned} \tag{4.24}$$

This means that the vectors corresponding to the matrices $(\partial^2 X - a_{i,1}\partial X - a_{i,2}\bar\partial X)$ and $(\bar\partial^2 X - a_{i,1}\partial X - a_{i,2}\bar\partial X)$, $i = 1, 2$ are normal to the surface determined by (4.12). Substituting (4.23) into equations (4.21) and solving the obtained linear systems we can determine $a_{i,l}$, $i, l = 1, 2$ which prove to be of the form (4.16).

Finally, the Gauss–Codazzi–Ricci (GCR) equations are the necessary and sufficient conditions for a local existence of a surface and are the compatibility conditions of the Gauss–Weingarten equations. In our case the GCR equations coincide with the $\mathbb{C}P^N$ sigma model equations (2.11) and are given in a matrix form by (4.17). So, with any solution f of the $\mathbb{C}P^N$ model we can associate a surface defined by (4.3). □

Making use of the expressions for the second derivatives of X, the induced metric and for the elements $a_{i,l}$ appearing in matrices A and B, we can write explicitly the second fundamental form of the surface in terms of the model

$$\begin{aligned}
\mathbf{II} &= (\partial^2 X)^\perp \,\mathrm{d}\xi^2 + 2(\partial\bar\partial X)^\perp \,\mathrm{d}\xi \,\mathrm{d}\bar\xi + (\bar\partial^2 X)^\perp \,\mathrm{d}\bar\xi^2\\
&= (\partial^2 X - a_{1,1}\partial X - a_{1,2}\bar\partial X)\,\mathrm{d}\xi^2 + 2(\partial\bar\partial X)\,\mathrm{d}\xi\,\mathrm{d}\bar\xi + (\bar\partial^2 X - a_{2,1}\partial X - a_{2,2}\bar\partial X)\,\mathrm{d}\bar\xi^2,
\end{aligned} \tag{4.25}$$

where the symbol $\perp$ denotes the normal part of matrices $\partial_i\partial_j X$ and the indices i, j stand for ξ or $\bar\xi$. The quantities $a_{i,l}$ are given by (4.16).

The mean curvature vector can also be expressed in terms of the model as follows:

$$\begin{aligned}
\mathbf{H} &= \frac{1}{g}(g_{\xi,\xi}(\bar\partial^2 X)^\perp - 2g_{\xi,\bar\xi}(\partial\bar\partial X)^\perp + g_{\bar\xi,\bar\xi}(\partial^2 X)^\perp)\\
&= \frac{1}{g}\{g_{\bar\xi,\bar\xi}[\partial^2 X - a_{1,1}\partial X - a_{1,2}\bar\partial X] - 2g_{\xi,\bar\xi}\partial\bar\partial X + g_{\xi,\xi}[\bar\partial^2 X - a_{2,1}\partial X - a_{2,2}\bar\partial X]\}.
\end{aligned} \tag{4.26}$$

The Willmore functional of a surface has the form

$$W = \int_\Omega |\mathbf{H}|^2 \sqrt{g}\,\mathrm{d}\xi\,\mathrm{d}\bar\xi. \tag{4.27}$$

When a solution f satisfying the CP^N model (2.11) is defined on the whole Riemannian sphere S^2 then the integral

$$\begin{aligned}\mathbf{Q} &= \frac{\mathrm{i}}{8\pi}\int_{S^2}\operatorname{tr}(P\cdot[\partial P,\bar\partial P])\,\mathrm{d}\xi\,\mathrm{d}\bar\xi \\ &= \frac{1}{8\pi}\int_{S^2}\frac{1}{f^\dagger f}(\partial f^\dagger P\bar\partial f-\bar\partial f^\dagger P\partial f)\,\mathrm{d}\xi\,\mathrm{d}\bar\xi\end{aligned} \tag{4.28}$$

is an invariant of the surface and is known as the topological charge of the model. If the integral (4.28) exists then it is an integer which characterizes globally the surface under consideration.

Summarizing, we can now state the following analogue of the Bonnet theorem, cf [46].

Corollary 2. *For the complex-valued function f satisfying the $\mathbb{C}P^N$ sigma model equations (2.11), the generalized Weierstrass formula for immersion (4.4), i.e.*

$$X:\Omega\ni(\xi,\bar\xi)\to X(\xi,\bar\xi)=\mathrm{i}\int_\gamma[\partial P,P]\,\mathrm{d}\xi+[\bar\partial P,P]\,\mathrm{d}\bar\xi, \tag{4.29}$$

describes a surface in $\mathfrak{su}(N+1)$. This surface is determined by its first and second fundamental forms (4.8) and (4.25) uniquely up to Euclidean motions.

Finally, it is worth noting that the method described above may be of use in the study of the elliptic periodic two-dimensional Toda lattice ($2DTL$) which is related to surfaces immersed in $\mathfrak{su}(N+1)$ Lie algebra [22]. The equations of $2DTL$ can be written in a matrix form as the zero curvature equations $\bar\partial A-\partial B=[A,B]$, formally identical with the Gauss–Codazzi–Ricci equation (4.17), where the two $(N+1)\times(N+1)$ matrices A and B are defined as follows:

$$A=-B^\dagger=\begin{pmatrix}\partial u_0 & 0 & 0 & \dots & 0 & U_{0,N}\\ U_{1,0} & \partial u_1 & 0 & \dots & 0 & 0\\ 0 & U_{2,1} & \partial u_2 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & 0 & \dots & \partial u_{N-1} & 0\\ 0 & 0 & 0 & \dots & U_{N,N-1} & \partial u_N\end{pmatrix}, \tag{4.30}$$

where for $i,j=0,\dots,N$, we set $u_i:\mathbb{C}\to\mathbb{R}$, $u_0+\dots+u_N=0$ and $U_{i,j}=\exp(u_i-u_j)$. It is known [22] that the zero-curvature equation (4.17) for matrices (4.30) implies the existence of a complex-valued function $F:\mathbb{C}\to\mathbf{SU}(N+1)$ such that

$$F^{-1}\partial F=A,\qquad F^{-1}\bar\partial F=B. \tag{4.31}$$

So, according to proposition 4, we can identify (4.31) with the complex tangent vectors (4.6) of the immersion (4.4). Hence the $2DTL$ equations can be viewed as being associated with the specific form (2.13) of the $\mathbb{C}P^N$ model. Establishing this link could be useful for determining certain geometric characteristics of surfaces corresponding to the elliptic periodic two-dimensional Toda lattice, but this point will not be considered here.

5. Immersions into the Lie algebra $\mathfrak{su}(2)$ arising from the $\mathbb{C}P^1$ model

In this section we sketch an application of the techniques developed in the previous sections to the case of the $\mathbb{C}P^1$ sigma model. This allows us to put the results obtained in the earlier publications [17, 19, 20] in a broader perspective, as well as to point out some further geometrical properties of surfaces obtained from this model.

The fields of the $\mathbb{C}P^1$ model in the notation of section 3 are given by $[z] = [f_0, f_1]$, where $z = (f_0, f_1) \in S^3 \subset \mathbb{C}^2$, but it is customary to replace here the homogeneous coordinates (f_0, f_1) by the affine coordinate $W = f_1/f_0$. The Euler–Lagrange equation (2.11) reduces to

$$\partial\bar{\partial}W - \frac{2\bar{W}}{1+|W|^2}\partial W\bar{\partial}W = 0 \tag{5.1}$$

and the matrix $\mathbb{K}$ of equation (2.14) is then given by

$$\mathbb{K} = \frac{1}{(1+|W|^2)^2}\begin{pmatrix} \bar{W}\bar{\partial}W - W\bar{\partial}\bar{W} & \bar{\partial}\bar{W} + \bar{W}^2\bar{\partial}W \\ -\bar{\partial}W - W^2\bar{\partial}\bar{W} & W\bar{\partial}\bar{W} - \bar{W}\bar{\partial}W \end{pmatrix}. \tag{5.2}$$

The 1-form $\mathrm{d}X = \mathrm{i}(\mathbb{K}^\dagger\,\mathrm{d}\xi + \mathbb{K}\,\mathrm{d}\bar{\xi})$ of the equation (4.1) defines an immersion into $\mathfrak{su}(2)$—to pass to the Euclidean space $\mathbb{R}^3$ we first compute its real and imaginary parts given by (4.2). These in turn can be expressed in terms of the Pauli matrices σ_i (cf equation (3.20)) as follows

$$\mathrm{d}X^1 = i\,\mathrm{d}X_1\,\sigma_2, \qquad \mathrm{d}X^2 = \mathrm{d}X_2\,\sigma_1 + \mathrm{d}X_3\,\sigma_3,$$

where the real-valued 1-forms $\mathrm{d}X_i$, $i = 1, 2, 3$, are given by

$$\begin{aligned} \mathrm{d}X_1 &= \frac{\mathrm{i}}{2(1+|W|^2)^2}\{-[\partial\bar{W} + \bar{W}^2\partial W + \partial W + W^2\partial\bar{W}]\,\mathrm{d}\xi \\ &\quad + [\bar{\partial}\bar{W} + \bar{W}^2\bar{\partial}W + \bar{\partial}W + W^2\bar{\partial}\bar{W}]\,\mathrm{d}\bar{\xi}\}, \\ \mathrm{d}X_2 &= \frac{1}{2(1+|W|^2)^2}\{[-\partial\bar{W} - \bar{W}^2\partial W + \partial W + W^2\partial\bar{W}]\,\mathrm{d}\xi \\ &\quad + [\bar{\partial}\bar{W} + \bar{W}^2\bar{\partial}W - \bar{\partial}W - W^2\bar{\partial}\bar{W}]\,\mathrm{d}\bar{\xi}\}, \\ \mathrm{d}X_3 &= \frac{1}{(1+|W|^2)^2}\{[W\partial\bar{W} - \bar{W}\partial W]\mathrm{d}\xi + [\bar{W}\bar{\partial}W - W\bar{\partial}\bar{W}]\mathrm{d}\bar{\xi}\}. \end{aligned} \tag{5.3}$$

This is the generalized Weierstrass formula for an immersion into $\mathbb{R}^3 \simeq \mathfrak{su}(2)$ we were aiming for. An interested reader may check that equations (5.3) yield the classical Weierstrass formula (1.2) under the substitution $W = f_1/f_0$, assuming holomorphicity of f_1, f_0.

Starting from a particular solution W of the $\mathbb{C}P^1$ sigma model equation (5.1) one constructs an immersion in $\mathbb{R}^3$ by the use of the formulae (5.3). The following is now readily obtained from proposition 4.

Corollary 3. *For the immersion given by equations (5.3) the coefficients of the induced metric are given by the following expressions:*

$$\begin{aligned} g_{11} &= \frac{|\partial W|^2 + |\bar{\partial}W|^2 + |\partial W - \bar{\partial}W|^2}{(1+|W|^2)^2}, \\ g_{22} &= \frac{|\partial W|^2 + |\bar{\partial}W|^2 + |\partial W + \bar{\partial}W|^2}{(1+|W|^2)^2}, \\ g_{12} &= \frac{2\,\mathrm{Im}(\partial W\partial\bar{W})}{(1+|W|^2)^2}. \end{aligned} \tag{5.4}$$

The complex form of the induced metric in this case is given by

$$g_{\xi,\xi} = -\frac{\partial W\partial\bar{W}}{(1+|W|^2)^2}, \qquad g_{\bar{\xi},\bar{\xi}} = -\frac{\bar{\partial}W\bar{\partial}\bar{W}}{(1+|W|^2)^2}, \qquad g_{\bar{\xi},\xi} = \frac{|\partial W|^2 + |\bar{\partial}W|^2}{(1+|W|^2)^2}. \tag{5.5}$$

For solutions of (5.1) which are defined over S^2, the function W can be only holomorphic or antiholomorphic (cf [48]). For holomorphic W equations (5.5) reduce to

$$g_{\xi,\xi} = g_{\bar{\xi},\bar{\xi}} = J = 0, \qquad g_{\bar{\xi},\xi} = \frac{|\partial W|^2}{(1+|W|^2)^2} \tag{5.6}$$

implying that the immersion is conformal. These relations, as shown earlier [17], imply also $g_{\bar{\xi},\bar{\xi}} = |Dz|^2$. The Gaussian curvature $\mathbf{K} = 1$, and the first and the second fundamental forms for the immersion are equal,

$$\mathbf{II} = \mathbf{I} = \frac{|\partial W|^2}{\left(1+|W|^2\right)^2}\,\mathrm{d}\xi\,\mathrm{d}\bar{\xi}. \tag{5.7}$$

Moreover, as shown by Kenmotsu in [27], the function W represents the complex Gauss map of the surface. Geometrically, all that means that solutions of the $\mathbb{C}P^1$ model (5.1) defined over S^2 parametrize the standardly immersed sphere in $\mathbb{R}^3$. This had already been shown in the case of instanton solutions of the $\mathbf{SO}(3)$ sigma model in [12].

6. Surfaces immersed in the $\mathfrak{su}(3)$ algebra

Here we apply our considerations to the $\mathbb{C}P^2$ model for which we construct the associated immersion of a surface $\mathbb{F}$ in $\mathbb{R}^8$ and compute some of its geometric characteristics. For the case of $N = 2$ we can pass from the representation $[z] = [f_0, f_1, f_2]$ to the inhomogeneous (affine) coordinates $W_1 = f_1/f_0$ and $W_2 = f_2/f_0$. Now the Euler–Lagrange equations (2.11) take the form

$$\begin{aligned}
&\partial\bar{\partial}W_1 - \frac{2\bar{W}_1}{A}\partial W_1\bar{\partial}W_1 - \frac{\bar{W}_2}{A}(\partial W_1\bar{\partial}W_2 + \bar{\partial}W_1\partial W_2) = 0,\\
&\partial\bar{\partial}W_2 - \frac{2\bar{W}_2}{A}\partial W_2\bar{\partial}W_2 - \frac{\bar{W}_1}{A}(\partial W_1\bar{\partial}W_2 + \bar{\partial}W_1\partial W_2) = 0,
\end{aligned} \tag{6.1}$$

where

$$A = 1 + |W_1|^2 + |W_2|^2. \tag{6.2}$$

As was noted in [20], the metric induced by the immersion $X(\xi,\bar{\xi})$ in $\mathbb{R}^8 \simeq \mathfrak{su}(3)$ is conformal for holomorphic solutions of the $\mathbb{C}P^2$ model defined over S^2 and is then given by

$$g_{\xi,\xi} = g_{\bar{\xi},\bar{\xi}} = 0, \qquad g_{\bar{\xi},\xi} = \frac{|\partial W_1|^2 + |\partial W_2|^2 + |W_1\partial W_2 - W_2\partial W_1|^2}{A^2}. \tag{6.3}$$

We shall write

$$g_{\bar{\xi},\xi} = \mathrm{e}^{\frac{1}{2}(u+\bar{u})}, \tag{6.4}$$

where u is a complex-valued function of $\xi, \bar{\xi} \in \mathbb{C}$, given by

$$u + \bar{u} = \ln\left\{\frac{1}{A^2}[|\partial W_1|^2 + |\partial W_2|^2 + |W_1\partial W_2 - W_2\partial W_1|^2]\right\}. \tag{6.5}$$

Under the above circumstances the following holds:

Proposition 4 (structural equations for holomorphic $\mathbb{C}P^2$ model)**.** *Any set of holomorphic solutions $(W_i, \bar{W}_i)$, $i = 1, 2$, of the $\mathbb{C}P^2$ sigma model equations (6.1) defined over S^2 such that the induced metric is nonzero in some neighbourhood of a regular point $p = (\xi_0, \bar{\xi}_0) \in \mathbb{C}$, determines a conformal parametrization of a surface $\mathbb{F}$ immersed in the $\mathfrak{su}(3)$ Lie algebra. Its moving frame on $\mathbb{F}$ can be written in terms of 3×3 traceless matrices and is of the form*

$$\eta = (\partial X, \bar{\partial}X, \eta_1, \ldots, \eta_6)^T, \tag{6.6}$$

where the complex tangent vectors to the surface $\mathbb{F}$ are given by

$$\partial X = -\frac{\mathrm{i}}{A^2}\begin{pmatrix} d_0 & \bar{W}_1 d_0 & \bar{W}_2 d_0\\ d_1 & \bar{W}_1 d_1 & \bar{W}_2 d_1\\ d_2 & \bar{W}_1 d_2 & \bar{W}_2 d_2\end{pmatrix} \qquad \bar{\partial}X = \frac{\mathrm{i}}{A^2}\begin{pmatrix} \bar{d}_0 & \bar{d}_1 & \bar{d}_2\\ W_1\bar{d}_0 & W_1\bar{d}_1 & W_1\bar{d}_2\\ W_2\bar{d}_0 & W_2\bar{d}_1 & W_2\bar{d}_2\end{pmatrix} \tag{6.7}$$

and where we have defined

$$\begin{aligned} d_0 &= -(\bar{W}_1\partial W_1 + \bar{W}_2\partial W_2),\\ d_1 &= (1+|W_2|^2)\partial W_1 - W_1\bar{W}_2\partial W_2,\\ d_2 &= (1+|W_1|^2)\partial W_2 - \bar{W}_1 W_2\partial W_1. \end{aligned} \tag{6.8}$$

Remark 2. The explicit expressions for the complex normals to this surface immersed in $\mathfrak{su}(3)$, in terms of W_1 and W_2, can be found in the appendix to [18].

Proof. Due to the normalization of the function X (given by equations (6.4) and (4.6)) we can express the moving frame $\eta = (\partial X, \bar{\partial} X, \eta_1, \ldots, \eta_6)^T$ on a surface $\mathbb{F}$ in terms of the adjoint $\mathfrak{su}(3)$ representation

$$\begin{cases} \partial X = \mathrm{e}^{u/2}\Phi^{-1}Y_-\Phi,\\ \bar{\partial} X = \mathrm{e}^{\bar{u}/2}\Phi^{-1}Y_+\Phi,\\ \eta_i = \Phi^{-1}S_{i+2}\Phi, \quad i = 1,\ldots,6, \end{cases} \tag{6.9}$$

where

$$Y_- = \frac{\mathrm{i}}{2}(S_1 - \mathrm{i}S_2) = \begin{pmatrix} 0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0 \end{pmatrix}, \qquad Y_+ = \frac{\mathrm{i}}{2}(S_1 + \mathrm{i}S_2) = \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{pmatrix}, \tag{6.10}$$

satisfy

$$(\Phi^{-1}Y_-\Phi)^\dagger = \Phi^{-1}Y_+\Phi.$$

Note that $\{Y_-, Y_+\}$ span over $\mathbb{R}$ the same space as $\{S_1, S_2\}$. Using the polar decomposition of the $\mathbf{SU}(3)$ group given in section 3, cf (3.13), a general $\mathbf{SU}(3)$ matrix Φ can be decomposed into a product of three $\mathbf{SU}(2)$ factors. Performing the multiplication in the expression (3.18) and setting $\lambda = \mathrm{e}^{\mathrm{i}\phi/2}$ and $\alpha = t$, we obtain

$$\Phi = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi}\cos t & -a_2\sin t & -b_2\sin t\\ a_1\sin t & a_1a_2\,\mathrm{e}^{-\mathrm{i}\varphi}\cos t - b_1\bar{b}_2 & b_2a_1\,\mathrm{e}^{-\mathrm{i}\varphi}\cos t + \bar{a}_2b_1\\ -\bar{b}_1\sin t & -a_2\bar{b}_1\,\mathrm{e}^{-\mathrm{i}\varphi}\cos t - \bar{a}_1\bar{b}_2 & -\bar{b}_1b_2\,\mathrm{e}^{-\mathrm{i}\varphi}\cos t + \bar{a}_1\bar{a}_2 \end{pmatrix} \in \mathbf{SU}(3), \tag{6.11}$$

where the complex-valued functions a_i, b_i of $\xi, \bar{\xi}$ satisfy $|a_i|^2 + |b_i|^2 = 1$ $i = 1, 2$, and φ, t are real-valued functions of $\xi, \bar{\xi} \in \mathbb{C}$. The requirement that the parametrization of a surface $\mathbb{F}$ is conformal implies the following relations:

$$\begin{aligned} &(\partial X, \partial X) = \mathrm{e}^{u}\,\mathrm{tr}(Y_-)^2 = 0, \qquad (\bar{\partial} X, \bar{\partial} X) = \mathrm{e}^{\bar{u}}\,\mathrm{tr}(Y_+)^2 = 0,\\ &(\partial X, \bar{\partial} X) = \mathrm{e}^{\frac{1}{2}(u+\bar{u})}\,\mathrm{tr}(Y_-\cdot Y_+) = \mathrm{e}^{\frac{1}{2}(u+\bar{u})}, \end{aligned} \tag{6.12}$$

and

$$\begin{aligned} (\partial X, \eta_i) &= \mathrm{e}^{u/2}\,\mathrm{tr}(Y_-\cdot S_{i+2}) = 0,\\ (\bar{\partial} X, \eta_i) &= \mathrm{e}^{\bar{u}/2}\,\mathrm{tr}(Y_+\cdot S_{i+2}) = 0,\\ (\eta_j, \eta_k) &= \mathrm{tr}(S_{j+2}\cdot S_{k+2}) = \delta_{jk}. \end{aligned} \tag{6.13}$$

Now we have to determine the form of a 8-parameter representation of the matrix Φ in terms of $W_i, \bar{W}_i$, compatible with the $\mathbb{C}P^2$ sigma model (6.1). Using the 3×3 projector matrix

$$P = \mathbf{1}_3 - \frac{1}{A}\begin{pmatrix} 1 & W_1 & W_2\\ \bar{W}_1 & W_1\bar{W}_1 & \bar{W}_1W_2\\ \bar{W}_2 & W_1\bar{W}_2 & \bar{W}_2W_2 \end{pmatrix}, \tag{6.14}$$

we can write the Euler–Lagrange equations (6.1) in the form of a conservation law (2.16) for the matrix

$$\mathbb{K} = \frac{1}{A}\begin{pmatrix} 0 & -\bar\partial \bar W_1 & -\bar\partial \bar W_2 \\ \bar\partial W_1 & \bar W_1 \bar\partial W_1 - W_1 \bar\partial \bar W_1 & \bar W_2 \bar\partial W_1 - W_1 \bar\partial \bar W_2 \\ \bar\partial W_2 & \bar W_1 \bar\partial W_2 - W_2 \bar\partial \bar W_1 & \bar W_2 \bar\partial W_2 - W_2 \bar\partial \bar W_2 \end{pmatrix} + \frac{\bar\rho}{A^2}\begin{pmatrix} 1 & \bar W_1 & \bar W_2 \\ W_1 & |W_1|^2 & W_1 \bar W_2 \\ W_2 & \bar W_1 W_2 & |W_2|^2 \end{pmatrix}, \tag{6.15}$$

where we have defined the following expression:

$$\rho = \bar W_1 \partial W_1 - W_1 \partial \bar W_1 + \bar W_2 \partial W_2 - W_2 \partial \bar W_2 \tag{6.16}$$

and the quantity A is given by equation (6.2). According to (2.14) and (4.6), the matrices ∂X and $\bar\partial X$ take the required form (6.7).

Let us note that to satisfy the compatibility condition for (6.9) (i.e. $\bar\partial\partial X = \partial\bar\partial X$) it is sufficient, in view of the conservation laws (2.13), to postulate that condition (4.6) holds for the matrix $\mathbb{K}$ given by (6.15). So, we can determine functions $a_i, b_i \in \mathbb{C}$ and $t, \varphi \in \mathbb{R}$, appearing in the matrix $\Phi \in \mathbf{SU}(3)$, in terms of W_i and $\bar W_i$, $i = 1, 2$. By a straightforward algebraic computation we get

$$\Phi = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi} A^{-1/2} & \bar W_1 \,\mathrm{e}^{\mathrm{i}\varphi} A^{-1/2} & \bar W_2 \,\mathrm{e}^{\mathrm{i}\varphi} A^{-1/2} \\ \mathrm{i}r^{-1}\,\mathrm{e}^{\mathrm{i}\varphi}(W_1 \partial W_1 + W_2 \partial W_2) A^{-1/2} & -\mathrm{i}r^{-1}\,\mathrm{e}^{\mathrm{i}\varphi} A^{-1/2} \bar d_1 & -\mathrm{i}r^{-1}\,\mathrm{e}^{\mathrm{i}\varphi} A^{-1/2} \bar d_2 \\ \mathrm{i}r^{-1}(W_1 \partial W_2 - W_2 \partial W_1)\,\mathrm{e}^{-2\mathrm{i}\varphi} & -\mathrm{i}r^{-1}\partial W_2 \,\mathrm{e}^{-2\mathrm{i}\varphi} & \mathrm{i}r^{-1}\partial W_1 \,\mathrm{e}^{-2\mathrm{i}\varphi} \end{pmatrix}, \tag{6.17}$$

where we have used the notation introduced in (6.8) and have set $r^2 = A^2 g_{\bar\xi,\xi}$.

Given the above form of the matrix Φ, the matrices Y_-, Y_+ and the S_{i+2}, $i = 1, \ldots, 6$ the moving frame (6.9) adopts the required forms (6.6) and (6.7). One can check directly that it satisfies the Gauss–Weingarten equations (4.12). In our case the corresponding GCR equations, which are the compatibility conditions for (4.12), coincide with the $\mathbb{C}P^2$ sigma model equations (6.1). Thus we have proved that any holomorphic solution of the $\mathbb{C}P^2$ model defined over S^2 gives a surface conformally immersed in $\mathbb{R}^8$ with the moving frame given by (6.6) and (6.7). □

7. Examples and applications for the $\mathbb{C}P^2$ model

Based on the results of the previous sections we can now construct certain classes of two-dimensional surfaces immersed in $\mathbb{R}^8$. For this purpose we use the $\mathbb{C}P^2$ sigma model defined over S^2. For this model all solutions of the Euler–Lagrange equations (6.1) are known [48]. If we require the finiteness of the action(2.9) they split into three classes: holomorphic (i.e. $W_i = W_i(\xi)$), antiholomorphic (i.e. $W_i = W_i(\bar\xi)$) and the mixed ones. The latter ones can be determined from either the holomorphic or antiholomorphic functions by the following procedure [48]:

Consider three arbitrary holomorphic functions, $g_i = g_i(\xi)$, $\bar\partial g_i = 0$, $i = 1, 2, 3$, and define for any pair of them the Wronskian functions

$$G_{ij} = g_i \partial g_j - g_j \partial g_i, \qquad i = 1, 2, 3 \tag{7.1}$$

Then one can check that the map $f = (f_1, f_2, f_3)$, where

$$f_i = \sum_{k \neq i}^{3} \bar g_k G_{ki}, \qquad i = 1, 2, 3 \tag{7.2}$$

is a solution of the $\mathbb{C}P^2$ sigma model, the so-called mixed one, and hence the ratios

$$W_1 = \frac{f_1}{f_3}, \qquad W_2 = \frac{f_2}{f_3} \tag{7.3}$$

satisfy equations (6.1).

An alternative approach starts with any antiholomorphic functions $\bar{g}_i = \bar{g}_i(\bar{\xi})$ and constructs functions f_i and consequently W_i as above but using $\bar{\partial}$ instead of ∂ in the definition of G_{ij}. It yields results which are complementary to those obtained by the first approach. Let us note here that the requirement of finite action (2.9) excludes solutions which admit Painlevé transcendent (i.e. all critical points are fixed independent of initial data), branch points or essential singularities.

Now, let us discuss some classes of surfaces immersed in $\mathfrak{su}(3)$ algebra which can be obtained directly by applying the Weierstrass representation (4.4). For the $\mathbb{C}P^2$ model (6.1), the matrix $\mathbb{K}$ in terms of W_i and $\bar{W}_i, i = 1, 2$ is given by (6.15). From equation (4.1) we obtain for the real and imaginary parts of the 1-form $\mathrm{d}X$ the expressions

$$\begin{aligned}
\mathrm{d}X^1 &= \frac{\mathrm{i}}{2}[(\mathbb{K}^\dagger - \bar{\mathbb{K}})\,\mathrm{d}\xi + (\mathbb{K} - \mathbb{K}^T)\,\mathrm{d}\bar{\xi}], \\
\mathrm{d}X^2 &= \frac{1}{2}[(\mathbb{K}^\dagger + \bar{\mathbb{K}})\,\mathrm{d}\xi + (\mathbb{K} + \mathbb{K}^T)\,\mathrm{d}\bar{\xi}].
\end{aligned} \tag{7.4}$$

Clearly, the matrices $\mathrm{d}X^1$ and $\mathrm{d}X^2$ are antisymmetric and symmetric, respectively, and hence can be decomposed in terms of the chosen basis in $\mathfrak{su}(3)$ given by (3.23)–(3.25) as follows

$$\begin{aligned}
\mathrm{d}X^1 &= \mathrm{d}X_2 S_2 + \mathrm{d}X_5 S_5 + \mathrm{d}X_6 S_6, \\
\mathrm{d}X^2 &= \mathrm{i}[\mathrm{d}X_1 S_1 + \mathrm{d}X_3 S_3 + \mathrm{d}X_4 S_4 + \mathrm{d}X_7 S_7 + \mathrm{d}X_8 S_8].
\end{aligned} \tag{7.5}$$

As a result of the decomposition (7.5), there exists eight real-valued functions $X_i(\xi, \bar{\xi}) i = 1, \ldots, 8$ which determine the generalized Weierstrass representation of surfaces associated with the $\mathbb{C}P^2$ model (6.1). Considering the off-diagonal entries of the matrices $\mathrm{d}X^1$ and $\mathrm{d}X^2$ we get

$$\begin{aligned}
\mathrm{d}X_1 &= \left[\frac{1}{A}(\partial\bar{W}_1 - \partial W_1) + \frac{\rho}{A^2}(\bar{W}_1 + W_1)\right]\mathrm{d}\xi + \left[\frac{1}{A}(\bar{\partial}W_1 - \bar{\partial}\bar{W}_1) + \frac{\bar{\rho}}{A^2}(\bar{W}_1 + W_1)\right]\mathrm{d}\bar{\xi}, \\
\mathrm{d}X_2 &= -\mathrm{i}\left\{\left[\frac{1}{A}(\partial W_1 + \partial\bar{W}_1) + \frac{\rho}{A^2}(\bar{W}_1 - W_1)\right]\mathrm{d}\xi + \left[-\frac{1}{A}(\bar{\partial}\bar{W}_1 + \bar{\partial}W_1) + \frac{\bar{\rho}}{A^2}(\bar{W}_1 - W_1)\right]\mathrm{d}\bar{\xi}\right\}, \\
\mathrm{d}X_5 &= -\mathrm{i}\left\{\left[\frac{1}{A}(\partial W_2 + \partial\bar{W}_2) + \frac{\rho}{A^2}(\bar{W}_2 - W_2)\right]\mathrm{d}\xi + \left[-\frac{1}{A}(\bar{\partial}\bar{W}_2 + \bar{\partial}W_2) + \frac{\bar{\rho}}{A^2}(\bar{W}_2 - W_2)\right]\mathrm{d}\bar{\xi}\right\},
\end{aligned} \tag{7.6}$$

$$\begin{aligned}
\mathrm{d}X_6 &= -\mathrm{i}\left\{\left[\frac{1}{A}(W_1\partial\bar{W}_2 - \bar{W}_2\partial W_1 - W_2\partial\bar{W}_1 + \bar{W}_1\partial W_2) + \frac{\rho}{A^2}(W_1\bar{W}_2 - \bar{W}_1 W_2)\right]\mathrm{d}\xi\right. \\
&\quad \left. + \left[\frac{1}{A}(\bar{W}_2\bar{\partial}W_1 - W_1\bar{\partial}\bar{W}_2 - \bar{W}_1\bar{\partial}W_2 + W_2\bar{\partial}\bar{W}_1) + \frac{\bar{\rho}}{A^2}(W_1\bar{W}_2 - \bar{W}_1 W_2)\right]\mathrm{d}\bar{\xi}\right\}, \\
\mathrm{d}X_7 &= \left[\frac{1}{A}(W_1\partial\bar{W}_2 - \bar{W}_2\partial W_1 + W_2\partial\bar{W}_1 - \bar{W}_1\partial W_2) + \frac{\rho}{A^2}(W_1\bar{W}_2 + \bar{W}_1 W_2)\right]\mathrm{d}\xi \\
&\quad + \left[\frac{1}{A}(\bar{W}_2\bar{\partial}W_1 - W_1\bar{\partial}\bar{W}_2 + \bar{W}_1\bar{\partial}W_2 - W_2\bar{\partial}\bar{W}_1) + \frac{\bar{\rho}}{A^2}(W_1\bar{W}_2 + \bar{W}_1 W_2)\right]\mathrm{d}\bar{\xi}, \\
\mathrm{d}X_8 &= \left[\frac{1}{A}(\partial\bar{W}_2 - \partial W_2) + \frac{\rho}{A^2}(\bar{W}_2 + W_2)\right]\mathrm{d}\xi + \left[\frac{1}{A}(\bar{\partial}W_2 - \bar{\partial}\bar{W}_2) + \frac{\bar{\rho}}{A^2}(\bar{W}_2 + W_2)\right]\mathrm{d}\bar{\xi}.
\end{aligned}$$

From the diagonal entries of the matrix $\mathrm{d}X^2$ we obtain

$$\mathrm{d}X_3 = 2\left\{\left[\frac{1}{A}(W_1\partial\bar{W}_1 - \bar{W}_1\partial W_1) + \frac{\rho}{A^2}|W_1|^2\right]\mathrm{d}\xi + \left[\frac{1}{A}(\bar{W}_1\bar{\partial}W_1 - W_1\bar{\partial}\bar{W}_1) + \frac{\bar{\rho}}{A^2}|W_1|^2\right]\mathrm{d}\bar{\xi}\right\},$$
$$\mathrm{d}X_4 = 2\left\{\left[\frac{1}{A}(W_2\partial\bar{W}_2 - \bar{W}_2\partial W_2) + \frac{\rho}{A^2}|W_2|^2\right]\mathrm{d}\xi + \left[\frac{1}{A}(\bar{W}_2\bar{\partial}W_2 - W_2\bar{\partial}\bar{W}_2) + \frac{\bar{\rho}}{A^2}|W_2|^2\right]\mathrm{d}\bar{\xi}\right\}. \tag{7.7}$$

Note that by virtue of the conservation law (2.16), the 1-forms (7.6) and (7.7) are the exact differentials of real-valued functions. The functions $X_j(\xi,\bar{\xi})$ $j = 1,\dots,8$ constitute the coordinates of the radius vector

$$\vec{X}(\xi,\bar{\xi}) = (X_1(\xi,\bar{\xi}),\dots,X_8(\xi,\bar{\xi})) \tag{7.8}$$

of a two-dimensional surface in $\mathbb{R}^8$. Thus, if the complex-valued functions $W_i, i = 1,2$ correspond to any solution of the $\mathbb{C}P^2$ sigma model (2.13), then we can use the generalized Weierstrass formulae (7.6) and (7.7) to construct a two-dimensional surface in $\mathbb{R}^8$ uniquely defined by this solution.

Let us note that in the limiting case when

$$W_i \to \frac{W}{\sqrt{2}}, \qquad i = 1,2, \tag{7.9}$$

or when we put $W_1 = 0$ or $W_2 = 0$, the Weierstrass formulae (7.6) and (7.7) reduce to formulae (5.3) describing the immersion in the $\mathbb{C}P^1$ case. These limits characterize some properties of solutions of both systems (5.1) and (6.1).

Now, let us discuss some classes of surfaces immersed in $\mathbb{R}^8$ which can be determined directly by applying the Weierstrass representation (7.6) and (7.7).

Example 1. As well known, the simplest case of solutions of the $\mathbb{C}P^2$ model is obtained by taking W_i to be analytic. In this case $\partial\bar{W}_i = 0$ and so many expressions in (7.6) and (7.7) vanish. In fact we get (with c.c. denoting the complex conjugate)

$$\begin{aligned}
\mathrm{d}X_1 &= \partial\left\{\frac{W_1+\bar{W}_1}{A}\right\}\mathrm{d}\xi + \text{c.c.}, & \mathrm{d}X_2 &= -\mathrm{i}\partial\left\{\frac{W_1-\bar{W}_1}{A}\right\}\mathrm{d}\xi + \text{c.c.},\\
\mathrm{d}X_3 &= 2\partial\left\{\frac{|W_1|^2}{A}\right\}\mathrm{d}\xi + \text{c.c.}, & \mathrm{d}X_4 &= 2\partial\left\{\frac{|W_2|^2}{A}\right\}\mathrm{d}\xi + \text{c.c.}.\\
\mathrm{d}X_5 &= -\mathrm{i}\partial\left\{\frac{W_2-\bar{W}_2}{A}\right\}\mathrm{d}\xi + \text{c.c.}, & \mathrm{d}X_6 &= -\mathrm{i}\partial\left\{\frac{\bar{W}_1W_2-\bar{W}_2W_1}{A}\right\}\mathrm{d}\xi + \text{c.c.},\\
\mathrm{d}X_7 &= \partial\left\{\frac{\bar{W}_1W_2+\bar{W}_2W_1}{A}\right\}\mathrm{d}\xi + \text{c.c.}, & \mathrm{d}X_8 &= \partial\left\{\frac{W_2+\bar{W}_2}{A}\right\}\mathrm{d}\xi + \text{c.c.}
\end{aligned} \tag{7.10}$$

These expressions can be easily integrated giving us, up to overall constants that can be added to any X_i:

$$\begin{aligned}
X_1 &= \left\{\frac{W_1+\bar{W}_1}{A}\right\}, & X_2 &= -\mathrm{i}\left\{\frac{W_1-\bar{W}_1}{A}\right\}, & X_3 &= 2\left\{\frac{|W_1|^2}{A}\right\},\\
X_4 &= 2\left\{\frac{|W_2|^2}{A}\right\} & X_5 &= -\mathrm{i}\left\{\frac{W_2-\bar{W}_2}{A}\right\}, & X_6 &= -\mathrm{i}\left\{\frac{\bar{W}_1W_2-\bar{W}_2W_1}{A}\right\},\\
X_7 &= \left\{\frac{\bar{W}_1W_2+\bar{W}_2W_1}{A}\right\}, & X_8 &= \left\{\frac{W_2+\bar{W}_2}{A}\right\}.
\end{aligned} \tag{7.11}$$

Note that in general we have a surface in $\mathbb{R}^8$. Using (6.3) it is very easy to calculate the curvature as we know that

$$g_{\bar{\xi},\xi} = A^{-2}\{|\dot{W}_1|^2 + |\dot{W}_2|^2 + |W_1\dot{W}_2 - W_2\dot{W}_1|^2\}, \qquad g_{\xi,\xi} = g_{\bar{\xi},\bar{\xi}} = 0, \tag{7.12}$$

where the dot denotes the differentiation with respect to ξ. Then the Gaussian curvature is given by

$$\mathbf{K} = -\frac{2}{g_{\bar{\xi},\xi}}\partial\bar{\partial}\ln g_{\bar{\xi},\xi}. \tag{7.13}$$

Now, by setting $W_2 = 0$ the above model is reduced to the $\mathbb{C}P^1$ case, with

$$X_1 = \frac{W_1 + \bar{W}_1}{1 + |W_1|^2}, \qquad X_2 = \mathrm{i}\frac{\bar{W}_1 - W_1}{1 + |W_1|^2}, \qquad X_3 = 2\frac{|W_1|^2}{1 + |W_1|^2} \tag{7.14}$$

and the remaining components $X_i = 0$ (for $i = 4, \ldots, 7$).

Note that in this case our surface is the surface of an appropriately located sphere. To see this note that

$$X_3 = 1 + \frac{1 - |W_1|^2}{1 + |W_1|^2}. \tag{7.15}$$

Then we have

$$X_1^2 + X_2^2 + (X_3 - 1)^2 = 1 \tag{7.16}$$

and so we see that all the points lie on the surface of a sphere of unit radius, centred at (0, 0, 1). Of course, the number of times this surface is covered depends on the degree of W_1, i.e. the topological charge of the map. This is, of course, consistent with (7.13), which gives a constant.

In the $\mathbb{C}P^2$ the situation is more complicated but also more can be said about the surface; i.e., for example, all points lie on the hyperellipsoid surface

$$X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 + 2X_6^2 + 2X_7^2 + X_8^2 = 2. \tag{7.17}$$

However, the Gaussian curvature is not necessarily constant. To see this we use (7.13) and consider very specific fields, namely,

$$W_1 = a\xi, \qquad W_2 = \xi^2, \qquad \text{where} \quad a \in \mathbb{R}. \tag{7.18}$$

Then

$$g_{\bar{\xi},\xi} = \frac{a^2 + 4|\xi|^2 + a^2|\xi|^4}{(1 + a^2|\xi|^2 + |\xi|^4)^2} \tag{7.19}$$

and so

$$\partial\bar{\partial}g_{\bar{\xi},\xi} = \frac{2(2 - a^4 - 2a^2(5 - a^4)|\xi|^2 + 4(2a^4 - 7)|\xi|^4 + 8a^2|\xi|^6 - (a^4 - 18)|\xi|^8 + 2a^2|\xi|^{10})}{(1 + a^2|\xi|^2 + |\xi|^4)^4}.$$

By further computation it can be checked that the Gaussian curvature $\mathbf{K}$ corresponding to this fields is not constant for any value of a.

Example 2. The simple mixed solution obtained by choosing $g_1 = 1$, $g_2 = \xi$, $g_3 = \xi^2$ in the formulae (7.1)–(7.3) gives us the following

$$W_1 = \frac{-\bar{\xi}(1+2|\xi|^2)}{\xi(2+|\xi|^2)}, \qquad W_2 = \frac{1-|\xi|^4}{\xi(2+|\xi|^2)}. \tag{7.20}$$

The Weierstrass representation (7.6), (7.7) can be integrated and it leads to the following expression for the immersion of our surface in $\mathbb{R}^8$ in polar coordinates (r, φ) by

$$\begin{aligned}
X_1(r,\varphi) &= \frac{-12r^4\cos 2\varphi}{(1+r^2+r^4)(1+4r^2+r^4)}, & X_2(r,\varphi) &= -\frac{12r^4\sin 2\varphi}{(1+r^2+r^4)(1+4r^2+r^4)},\\
X_3(r,\varphi) &= \frac{-4(4r^6+6r^4+9r^2+2)}{(1+r^2+r^4)(1+4r^2+r^4)}, & X_4(r,\varphi) &= \frac{12r^2(1+r^4)}{(1+r^2+r^4)(1+4r^2+r^4)},\\
X_5(r,\varphi) &= -\frac{2(r^8+7r^6-r^2-1)\sin\varphi}{r(1+r^2+r^4)(1+4r^2+r^4)}, & X_6(r,\varphi) &= -\frac{-4(r^8-2r^6-4r^2-1)\sin\varphi}{r(1+r^2+r^4)(1+4r^2+r^4)},\\
X_7(r,\varphi) &= \frac{-4(r^8-2r^6-4r^2-1)\cos\varphi}{r(1+r^2+r^4)(1+4r^2+r^4)}, & X_8(r,\varphi) &= \frac{-2(r^8+7r^6-r^2-1)\cos\varphi}{r(1+r^2+r^4)(1+4r^2+r^4)}.
\end{aligned} \tag{7.21}$$

The curvatures can be calculated, but the expressions are rather involved, so we omit them here.

Example 3. Another interesting class of mixed solutions of the $\mathbb{C}P^2$ model is given by

$$W_1 = \frac{\xi+\bar{\xi}}{1-|\xi|^2}, \qquad W_2 = \frac{\bar{\xi}-\xi}{1-|\xi|^2}. \tag{7.22}$$

In this case the system of Euler–Lagrange equations (6.1) simplifies considerably and therefore the Weierstrass formulae (7.6), (7.7) can be easily integrated. In polar coordinates, setting $r = \mathrm{e}^{\vartheta}$, we obtain

$$\begin{aligned}
X_2(\vartheta,\varphi) &= \mathrm{e}^{-\vartheta}\tanh\vartheta\,\sin\varphi, & X_8(\vartheta,\varphi) &= \mathrm{e}^{-\vartheta}\tanh\vartheta\,\cos\varphi,\\
X_7(\vartheta,\varphi) &= \mathrm{e}^{-\vartheta}\,\mathrm{sech}\,\vartheta, & X_1 &= X_3 = X_4 = X_5 = X_6 = 0.
\end{aligned} \tag{7.23}$$

This describes a surface of revolution which is contained in a subspace of dimension 3 (see figure 1). The first and second fundamental forms are

$$\begin{aligned}
\mathbf{I} &= \frac{1}{r^2(1+r^2)^2}[r^{-2}(r^4+6r^2+1)\,\mathrm{d}r^2 + (r^2-1)^2\,\mathrm{d}\varphi^2],\\
\mathbf{II} &= \frac{4}{(1+r^2)^2(r^4+6r^2+1)^{1/2}}[(r^2+3)\,\mathrm{d}r^2 + r^2(r^2-1)\,\mathrm{d}\varphi^2].
\end{aligned} \tag{7.24}$$

The Gaussian and mean curvature are

$$\mathbf{K} = \frac{16r^8(r^2+3)}{(r^4+6r^2+1)^2(r^2-1)}, \qquad \mathbf{H} = \frac{r^4(r^4+4r^2-1)}{(r^4+6r^2+1)^{3/2}(r^2-1)}. \tag{7.25}$$

Since the curvatures are not constant, the surface cannot be obtained from the $\mathbb{C}P^1$ model.

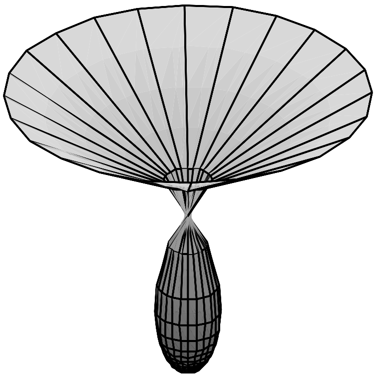

Figure 1. The surface associated with solution (7.23).

8. Concluding remarks and prospects for future developments

There are reasons to expect that the association between the Weierstrass representation of surfaces immersed in $\mathfrak{su}(N+1)$ Lie algebras and the solutions of the Euclidean two-dimensional CP^N sigma models, described in this paper, can be found also for more general sigma models. A good object of the investigation in this direction are the complex Grassmannian sigma models which take values on symmetric spaces $\mathbf{SU}(m+n)\,/\mathbf{S}(\mathbf{U}(m)\times\mathbf{U}(n))$. These models share many important common properties with the CP^N models considered here. They possess an infinite number of conserved quantities, as well as infinite-dimensional dynamical symmetries which generate the Kac–Moody algebra. The Grassmannian sigma model equations, just like those of the $\mathbb{C}P^N$ models, have a Hamiltonian structure and complete integrability with a well-formulated linear spectral problem. Many classes of solutions of these equations are known, see e.g. [48]; they can be expressed in terms of holomorphic functions and functions obtained from them by a procedure which is a generalization of the transformation which generates all solutions of the $\mathbb{C}P^N$ models.

The complex Grassmannian sigma models in two Euclidean dimensions are defined in terms of fields

$$g = g(\xi, \bar{\xi}) \in \mathbf{SU}(N+1), \tag{8.1}$$

where $\xi = \xi^1 + \mathrm{i}\xi^2$, taking values in the complex Grassmann manifold $\mathbf{SU}(N+1)\,/\mathbf{S}(\mathbf{U}(m)\times\mathbf{U}(n))$, where $N+1 = m+n$. By decomposing g into two blocks

$$g = (X, Y), \qquad X = (z_1, \ldots, z_m), \qquad Y = (z_{m+1}, \ldots, z_{N+1}),$$

where z_i are $(N+1)$-component orthonormal column vectors,

$$z_i^\dagger \cdot z_k = \delta_{ik} \tag{8.2}$$

we define the projector matrix P ($P^\dagger = P$, $P^2 = P$) as

$$P = XX^\dagger = \sum_{l=1}^{m} z_l z_l^\dagger. \tag{8.3}$$

In general, it has a higher rank than the corresponding matrix for the $\mathbb{C}P^N$ model. However, the equation of motion in terms of P in this case has the same form as (2.4) and is obtained by minimizing the action of the Lagrangian

$$L = \mathrm{tr}[(D_\mu X)^\dagger \cdot (D_\mu X)], \tag{8.4}$$

where $D_\mu X$ is the covariant derivative for X,

$$D_\mu X = \partial_\mu X - X(X^\dagger \cdot \partial_\mu X). \tag{8.5}$$

The above fact implies that our method can be successfully used for constructing surfaces associated with the complex Grassmannian sigma models. The question of the diversity and complexity of these surfaces, however, remains open and has to be answered in further work.

In this paper we have shown how to generalize the old idea of Enneper [11] and Weierstrass [45] in connection with the $\mathbb{C}P^N$ sigma models and their group properties. We have found the structural equations of surfaces immersed in $\mathfrak{su}(N+1)$ Lie algebras and expressed them in terms of any solution of the $\mathbb{C}P^N$ model. The most important advantage of the presented method is that it is quite general. In constructing surfaces we proceeded directly from the given $\mathbb{C}P^N$ model, without referring to any additional considerations. Another important advantage of our method is that, due to the conservation laws of the $\mathbb{C}P^N$ model, the obtained expressions for surfaces are given at least in the form of quadratures.

We have discussed in detail the geometrical aspects of the constructed surfaces. Namely, we have demonstrated through the use of moving frame that one can derive, via the $\mathbb{C}P^N$ models, the first and the second fundamental forms of a given surface as well as relations between them as expressed in the Gauss–Weingarten and the Gauss–Codazzi–Ricci equations. We have illustrated the proposed method of constructing surfaces in the case of low dimensional $\mathfrak{su}(N+1)$ Lie algebras.

A systematic application of the group theory makes it possible to obtain a large number of particular solutions of the $\mathbb{C}P^N$ equations and associated surfaces in $\mathbb{R}^{N(N+2)}$. A question arises whether these solutions, corresponding to specific boundary conditions, are actually observable in nature. The answer depends to a large degree on their stability. Stable solutions should be observable and should also provide the starting point for perturbative calculations. These should in turn provide good approximative solutions relevant for situations in which the group-theoretical solutions no longer apply. In this context another question arises, namely what physical insight one gains from exact analytic expressions for surfaces. A particular answer is that they show up qualitative features that might be difficult to detect numerically. We hope that our approach and our results may be useful in applications to the study of surfaces which arise in physics, chemistry and biology, by providing explicit models in situations which have been well investigated experimentally but for which the theory is not yet well developed. Further exploration of relations between various properties of harmonic maps $S^2 \to \mathbb{C}P^N$ and properties of surfaces in planed for future work.

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